

Dynamic Response of a Double Beam System with Nonlinear Variable-Stiffness Viscoelastic Layer under a Moving Load



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ABSTRACT

This study investigated the dynamic response of a simply supported double beam system interconnected through a viscoelastic nonlinear variable stiffness layer with a Winkler support beneath the lower beam and traversed by a moving force. The governing equations were formulated to account for the nonlinear variable stiffness, viscoelastic coupling, foundation stiffness, and moving force velocity. The Fourier sine integral transform was employed to simplify the spatial dependence of the governing equations, while the differential transform method was applied to obtain a semi-analytical representation of the dynamic response. Parametric analyses were conducted to examine the effects of moving force velocity, viscoelasticity, nonlinear variable stiffness, interlayer stiffness, and Winkler foundation stiffness on the responses of the upper and lower beams. The results show that increasing the moving force velocity increases the response of both beams through enhanced dynamic excitation and load transmission, whereas increasing the viscoelastic parameter reduces both responses through increased energy dissipation. Higher nonlinear variable stiffness and interlayer stiffness drive the upper beam towards rigid coupling but increase the lower beam response through enhanced load transfer. Increasing the Winkler foundation stiffness promotes rigid coupling in the upper beam and brings the lower beam closer to the rigid coupling condition. The findings provide insight into the influence of coupling and foundation parameters on the vibration behaviour of nonlinear double beam systems subjected to moving forces.

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INTRODUCTION

Beams are fundamental load carrying members in civil, mechanical, aerospace and transportation structures. Their relatively slender geometry makes them particularly suitable for resisting transverse loading, while their simple structural representation allows a wide range of practical systems to be modelled using beam theories (Frýba, 1972). Bridges, railway tracks, crane structures, guideways, pipelines, aircraft components and other transportation related structures are typical examples in which beam like members are exposed to loads that vary with time and position (Frýba, 1999; Tabejieu et al., 2017). The dynamic analysis of such structures becomes particularly important when the applied load is mobile, since the continuous movement of the load introduces a coupled spatial and temporal variation in the structural response. The resulting vibration may become pronounced at high speeds and can consequently influence the dynamic stresses and

serviceability of the structure (Frýba, 1972; Abu-Hilal, 2006; Li et al., 2016).

The vibration analysis of single beams has been well established and comprehensive results reported in various contributions (Oni and Awodola 2005; Hryniewicz 2011; Usman et al. 2018). The increasing demand for structures with improved stiffness and load carrying capability has also encouraged the use of coupled beam systems (Oniszcuk, 2003; Stojanović & Kozić, 2012). A coupled double beam system, which consists of two parallel beams that interact through an intermediate layer, provides another load transfer path and modifies the vibration characteristics of the overall system. Compared to single beams, double beams exhibit higher strength and stiffness as well as better vibration absorption properties for the same weight (Mirzabeigy et al., 2017; Singh & Srilatha, 2018; Lo Feudo et al., 2019; Zhao et al., 2020). The mechanical behaviour of the connecting layer is therefore an important part of the

dynamic model since it governs the transfer of transverse motion and forces between the two beams.

The dynamic behaviour of double beam structures subjected to moving loads has received considerable attention. Abu-Hilal (2006) investigated the response of two simply supported Euler–Bernoulli beams continuously connected by a viscoelastic layer and subjected to a moving constant load. The study demonstrated the influence of load velocity, interlayer stiffness and damping on the dynamic responses of the two beams. Subsequent investigations have considered other boundary conditions, elastic and viscoelastic connecting layers, and different forms of moving excitation. Hammed et al., (2020) examined an elastically restrained double beam system subjected to a concentrated moving mass and explored modified Struble and differential transform techniques in obtaining the transverse response of the two beams. Other research has extended double Euler-Bernoulli beam models to include structural damage, moving vehicles and vibration control devices, further demonstrating the importance of accounting for interaction between the beams and the moving excitation.

Despite these developments, most existing formulations have concentrated on relatively simple interconnecting mechanisms or have treated the moving excitation using a prescribed force representation while neglecting the effect of the nonlinear variable stiffness layer. For example, the classical double Euler–Bernoulli beam model developed by Abu-Hilal (2006) and Li et al. (2016) considered identical simply supported beams connected through a continuously distributed viscoelastic layer and subjected to a moving constant load.

Nonlinear elastic layers are widely used in engineering applications where vibration isolation, shock absorption, and energy dissipation are essential. In the design of railway tracks, for instance, nonlinear elastic layers are used to reduce the transmission of vibrations from moving trains to the track foundation (Kozioł and Pileck, 2018; Adali, 2023). This is achieved by placing elastic

pads or under sleeper pads beneath the railway sleepers, allowing for vibration reduction, noise isolation, and track stability. The nonlinearity and spatial variability of the interconnecting layer introduce additional mechanisms into the dynamic interaction between the beams. Changes in the stiffness characteristics of the layer can alter the degree of coupling, the amount of load transferred to the secondary beam and the amplitude of the transverse vibration (Kozioł and Pileck, 2021). The nonlinearity of these layers ensures that they provide effective isolation over a wide range of train speeds and axle loads (Li et al., 2016; Mirzabeigy and Madoliat, 2019).

In view of this, this study investigates the combined influence of nonlinear variable-stiffness and viscoelastic coupling on the dynamic response of a double Euler-Bernoulli beam system resting on a Winkler foundation under a moving load, which has not been accounted for in the existing literature. The coupled equations of motion are formulated for the two beams, with the nonlinear and viscoelastic characteristics of the interconnecting layer incorporated into the model. This study aims to investigate the dynamic response of a simply supported double Euler-Bernoulli beam system interconnected by a nonlinear variable-stiffness viscoelastic layer and subjected to a moving concentrated load.

MATERIALS AND METHODS

The structural model consists of two identical, parallel, finite Euler-Bernoulli beams of length L and mass per unit length μ , which are assumed to be uniform, undamped, and simply supported at both ends. The beams exhibit linear elastic material behaviour and are interconnected by a nonlinear variable stiffness viscoelastic layer, while an additional elastic foundation supports the lower beam. The cross sections are assumed to remain rigid and uniform along the beam length, with the transverse motion confined to a single plane.

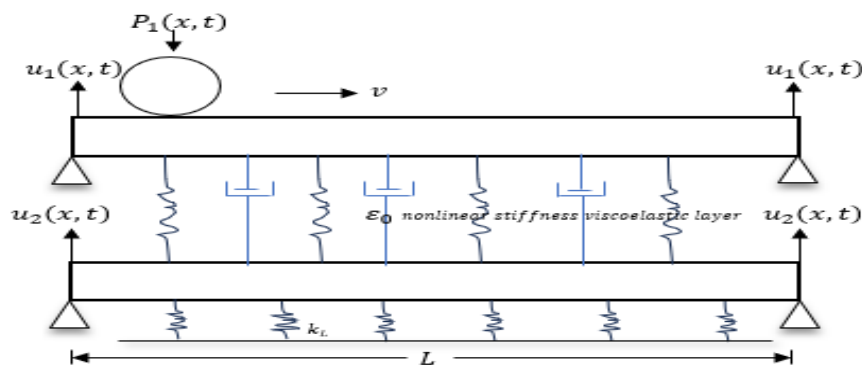


Figure 1: A double Beam System Subjected to a Moving Load (Li et al. 2016)

The structural model is under a dynamic concentrated moving load, $P_1(x, t)$, with a constant velocity, v . The dynamic transverse responses of the primary and secondary beams are denoted as $u_1(x, t)$ and $u_2(x, t)$, respectively. The problem under consideration is characterised by a pair of coupled fourth order partial differential equations which are expressed as:

$$EI_1 \frac{\partial^4 u_1(x, t)}{\partial x^4} + \mu \frac{\partial^2 u_1(x, t)}{\partial t^2} + (k_0(1 + k_2 x^2) + \varepsilon_0 \frac{\partial}{\partial t}) [u_1(x, t) - u_2(x, t)] = P_1(x, t) \quad (1)$$

$$EI_2 \frac{\partial^4 u_2(x, t)}{\partial x^4} + \mu \frac{\partial^2 u_2(x, t)}{\partial t^2} + (k_0(1 + k_2 x^2) + \varepsilon_0 \frac{\partial}{\partial t}) [u_2(x, t) - u_1(x, t)] + k_L u_2(x, t) = 0 \quad (2)$$

A constant force R_0 traversing the length of the beam from left to right with a steady speed v , is well observed such that the load $P_1(x, t)$ is defined as,

$$P_1(x, t) = R_0 \delta(x - vt) \quad (3)$$

Where $\delta(x - vt)$ is the Dirac delta function which is expressed as,

$$\delta(x - vt) = \frac{1}{L} + \frac{2}{L} \sum_{n=0}^{\infty} \cos \frac{n\pi vt}{L} \cos \frac{n\pi x}{L} \quad (4)$$

Where EI_1 and EI_2 are the flexural rigidity of the primary and secondary beams respectively. The non-linear stiffness, elastic and damping parameters of the viscoelastic foundation layer are represented by k_2, k_0, ε_0 respectively.

For the system at hand, the boundary conditions considered at each end of the beams are, for $i = 1, 2$:

$$u_i(0, t) = 0 = u_i(L, t) \quad (5)$$

$$\frac{\partial^2 u_i(0, t)}{\partial x^2} = 0 = \frac{\partial^2 u_i(L, t)}{\partial x^2}; \quad i = 1, 2 \quad (6)$$

Initial conditions are:

$$u_i(x, 0) = 0 = \frac{\partial u_i(x, 0)}{\partial t}, \quad i = 1, 2 \quad (7)$$

Solution Procedure

Finite Fourier Sine Integral Sine Transform

A finite Fourier sine integral transformation method was initially deployed for the transformation process. This method was chosen due to its suitability in the analysis of moving loads, as established by previous studies (Gbadeyan & Oni, 1995; Hammed et al., 2025; Mohammadi & Nasirshoabi, 2015; Rajib et al., 2012). Therefore, recalling the finite sine Fourier Sine assumed solution of the form,

$$\bar{u}_m(q, t) = \int_0^L u_m(x, t) V_q(x); \quad q = 1, 2, 3, \dots; \quad m = 1, 2. \quad (8)$$

The corresponding inverse formulae is,

$$u_m(x, t) = \frac{2}{L} \sum_{n=1}^{\infty} \bar{u}_m(q, t) V_q(x), \quad m = 1, 2 \quad (9)$$

Where

$$V_q(x) = \sin \frac{\lambda_q}{L} x. \quad (10)$$

Applying Equation (8) into Equations (1) and (2) appropriately with some simplifications, the resulting equations are,

$$\bar{\bar{u}}_1(q, t) + \omega_q^2 \bar{u}_1(q, t) + \frac{k}{\mu} [\bar{u}_1(q, t) - \bar{u}_2(q, t)]$$

$$+ \frac{\varepsilon_0}{\mu} [\bar{\bar{u}}_1(q, t) - \bar{\bar{u}}_2(q, t)] = \frac{M_L}{\mu L} V_q(vt) \quad (11)$$

$$\bar{\bar{u}}_2(q, t) + \omega_q^2 \bar{u}_2(q, t) + \frac{k}{\mu} [\bar{u}_2(q, t) - \bar{u}_1(q, t)]$$

$$+ \frac{\varepsilon_0}{\mu} [\bar{\bar{u}}_2(q, t) - \bar{\bar{u}}_1(q, t)] + \frac{k_L}{\mu} \bar{u}_2(q, t) = 0 \quad (12)$$

Where

$$\omega_q^2 = \frac{q^4 \pi^4 EI}{L^4 \mu}, \quad \omega_r^2 = \frac{r^4 \pi^4 EI}{L^4 \mu}, \quad k = 1 + k_2 d_2, \quad d_2 = \frac{L^2}{3} - \frac{L^2}{4\lambda_q^2} \quad (13)$$

Equations (11) and (12) are decoupled intermediate results, and thereby requiring further analysis to derive the desired solutions. Thus, a semi analytical method called differential transform method was introduced.

Differential Transform Method (DTM)

The basic concept of differential transform is considered for an analytic function $u_m(q, t)$ involving continuous derivative desired in a finite domain. Thus,

$$\bar{u}_m(k) = \frac{1}{k!} \left[\frac{d^k u_m(q, t)}{dt^k} \right]_{t=t_0} \quad (14)$$

Where $u_m(q, t)$ the main function and the spectrum is $\bar{u}_m(k)$ is the modified function which can shortly be written as T-function. The technique of differential inverse (DIT) of $\bar{u}_m(k)$ as represented in Hosseinzadeh and Salehpour (2013)

$$u_m(q, t) = \sum_{k=0}^{\infty} \bar{u}_m(k) (t - t_0)^k \quad (15)$$

In view of Equations (12) and (13), Equation (15) can be expressed as,

$$u_m(q, t) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} \left[\frac{d^k \bar{u}_m(q, t)}{dt^k} \right] \quad (16)$$

And when the value of $t_0 = 0$, Equation (16) becomes,

$$u_m(q, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k \bar{u}_m(q, t)}{dt^k} \right]_{t=t_0} \quad (17)$$

Hence,

$$u_m(q, t) = \sum_{k=0}^{\infty} \bar{u}_m(k) t^k \quad (18)$$

Equation (18) indicates that the idea of differential technique is obtained from Taylor series scheme. The key distinction between Taylor series scheme and differential transform technique is that the first techniques call for simplifications of higher order differentials that are quite rarely formidable, while the other requires iterative schemes. In the practical experience in relations to the current analyses, the function $u_m(q, t)$ should be a limited series so that Equation (18) can be written as,

$$u_m(q, t) = \sum_{k=0}^m \bar{u}_m(k) t^k \quad (19)$$

Thus, the series $\sum_{k=m+1}^{\infty} \bar{u}_m(k) t^k$ in Equation (19) is considered to be negligibly unimportant so that the values of m are determined by the natural frequency convergence in the analyses. The fundamental theorems in the modification of the dynamic equations and the boundary constraints as employed in Hosseinzadeh and Salehpour (2013) are used in this study as shown in Table 1.

Table 1: Some Theorems of DTM for Equations of Motion

Original function	T-function
$u(t) = U(t) \pm v(t)$	$\bar{u}(k) = \bar{U}(k) \pm \bar{V}(k)$
$u(t) = cU(t)$	$\bar{u}(k) = c\bar{U}(k)$
$u(t) = \frac{dU(t)}{dt}$	$\bar{u}(k) = (k+1)\bar{U}(k+1)$
$u(t) = \frac{d^n U(t)}{dt^n}$	$\bar{u}(k) = (k+1)(k+2) \dots (k+n-1)(k+n)\bar{U}(k+n)$
$u(t) = U(t)v(t)$	$\bar{u}(k) = \sum_{n=0}^k \bar{U}(n)\bar{V}(k-n)$
$u(t) = U(t)v(t)$	$\bar{u}(k) = \sum_{n=0}^k \sum_{r=0}^{k-n} \bar{U}(n)\bar{V}(r)\bar{Y}(k-n-r)$
$u(t) = U(t) \int_0^L v(x)dx$	$\bar{u}(k) = \sum_{n=0}^k \bar{U}(k-n) \frac{\bar{V}(n-1)}{n} k \geq 1$
$u(t) = t^m$	$\bar{u}(k) = \delta(k-m) = \begin{cases} 1, k=m \\ 0, k \neq m \end{cases}$
$u(t) = \sin at$	$\bar{u}(k) = \frac{1}{k!} a^k \sin\left(\frac{k\pi}{2}\right)$
$u(t) = \cos at$	$\bar{u}(k) = \frac{1}{k!} a^k \cos\left(\frac{k\pi}{2}\right)$
$u(t) = \sinh at$	$\bar{u}(k) = \frac{1}{2k!} [(a)^k - (-a)^k]$
$u(t) = \cosh at$	$\bar{u}(k) = \frac{1}{2k!} [(a)^k + (-a)^k]$

In view of Table 1, Equations (11) and (12) becomes

$$\bar{u}_1(k+2) = \frac{1}{(k+1)(k+2)} \left[P_1 \frac{1}{k!} \left(\frac{\lambda_q v}{L} \right)^k \sin\left(\frac{k\pi}{2}\right) - \omega_q^2 \bar{u}_1(k) - N_1 [\bar{u}_1(k) - \bar{u}_2(k)] - N_2(k+1) [\bar{u}_1(k+1) - \bar{u}_2(q, t)] \right] \quad (20)$$

$$\bar{u}_2(k+2) = \frac{1}{(k+1)(k+2)} [-\omega_q^2 \bar{u}_2(k) - N_1 [\bar{u}_2(k) - \bar{u}_1(k)] - N_2 [\bar{u}_2(k+1) - \bar{u}_1(k+1)] - N_3 \bar{u}_2(k)] \quad (21)$$

$$\text{Where } N = \frac{k}{\mu}, \quad N_2 = \frac{\varepsilon_0}{\mu}, \quad N_3 = \frac{k_L}{\mu}, \quad P_1 = \frac{M_L}{\mu L}.$$

The transformed initial conditions at $t_0 = 0$ are,

$$\bar{u}_1(0) = \bar{u}_1(1) = 0 \quad (22)$$

The $k \geq 0$ values were taken into consideration in Equations (20) and (21) and, having observed the derived initial conditions in Equations (22) and (23).

Hence, for: $k = 0$,

$$\bar{u}_1(2) = 0 \quad (23)$$

$$\bar{u}_2(2) = 0 \quad (24)$$

$k = 1$;

$$\bar{u}_1(3) = \frac{P_1}{3!} \left(\frac{\lambda_q v}{L} \right) \quad (25)$$

$$\bar{u}_2(3) = 0 \quad (26)$$

$k = 2$;

$$\bar{u}_1(4) = -\frac{N_2 P_1}{4!} \left(\frac{\lambda_q v}{L} \right) \quad (27)$$

$$\bar{u}_2(4) = \frac{N_2 P_1}{4!} \left(\frac{\lambda_q v}{L} \right) \quad (28)$$

$k = 3$;

$$\bar{u}_1(5) = -\frac{P_1}{5!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 + \omega_q^2 + N - 2N_2^2 \right] \quad (29)$$

$$\bar{u}_2(5) = \frac{P_1}{5!} \left(\frac{\lambda_q v}{L} \right) [N - 2N_2^2] \quad (31)$$

$k = 4$;

$$\bar{u}_1(6) = \frac{N_2 P_1}{6!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 + 2\omega_q^2 + 4N - 4N_2^2 \right] \quad (32)$$

$$\bar{u}_2(6) = -\frac{N_2 P_1}{6!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 + 2\omega_q^2 - 4N + 4N_2^2 - N_3 \right] \quad (33)$$

$$\bar{u}_1(7) = \frac{P_1}{7!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 \left(\left(\frac{\lambda_q v}{L} \right)^2 + \omega_q^2 + N - 2N_2^2 \right) + \omega_q^2 (\omega_q^2 + 2N - 6N_2^2) + N^2 - 2NN_2^2 + N_2^2 N_3 \right] \quad (34)$$

$$\bar{u}_2(7) = \frac{P_1}{7!} \left(\frac{\lambda_q v}{L} \right) \left[N_2^2 \left(2 \left(\left(\frac{\lambda_q v}{L} \right)^2 + \omega_q^2 \right) + N_3 + 4\omega_q^2 + 4N \right) - N \left(2N + N_3 + \left(\frac{\lambda_q v}{L} \right)^2 \right) \right] \quad (35)$$

By applying Equations (24) – (35) into Equations (19) for which $m = 1, 2$ respectively, the obtained modal amplitudes were in turn used in Equation (9) to yield the following results:

$$u_1(x, t) = \frac{2}{L} \sum_{q=1}^{\infty} \left[\frac{P_1}{3!} \left(\frac{\lambda_q v}{L} \right) t^3 - \frac{N_2 P_1}{4!} \left(\frac{\lambda_q v}{L} \right) t^4 - \frac{P_1}{5!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 + \omega_q^2 + N - 2N_2^2 \right] t^5 + \frac{N_2 P_1}{6!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 + 2\omega_q^2 + 4N - 4N_2^2 \right] t^6 + \frac{P_1}{7!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 \left(\left(\frac{\lambda_q v}{L} \right)^2 + \omega_q^2 + N - 2N_2^2 \right) + \omega_q^2 (\omega_q^2 + 2N - 6N_2^2) + N^2 - 2NN_2^2 + N_2^2 N_3 \right] t^7 + \dots \right] \left[\sin \frac{\lambda_q x}{L} \right] \quad (36)$$

$$u_2(x, t) = \frac{2}{L} \sum_{q=1}^{\infty} \left[\frac{N_2 P_1}{4!} \left(\frac{\lambda_q v}{L} \right) t^4 + \frac{P_1}{5!} \left(\frac{\lambda_q v}{L} \right) [N - 2N_2^2] t^5 - \frac{N_2 P_1}{6!} \left(\frac{\lambda_q v}{L} \right) \left[\left(\frac{\lambda_q v}{L} \right)^2 + 2\omega_q^2 - 4N + 4N_2^2 - N_3 \right] t^6 + \frac{P_1}{7!} \left(\frac{\lambda_q v}{L} \right) \left[N_2^2 \left(2 \left(\left(\frac{\lambda_q v}{L} \right)^2 + \omega_q^2 \right) + N_3 + 4\omega_q^2 + 4N \right) - N \left(2N + N_3 + \left(\frac{\lambda_q v}{L} \right)^2 \right) \right] t^7 + \dots \right] \left[\sin \frac{\lambda_q x}{L} \right] \quad (37)$$

Thus, Equations (36) and (37) are the desired dynamic response equations of the simply supported upper and

lower Euler-Bernoulli beams with nonlinear interconnected variable layer traversed by a concentrated moving force.

RESULTS AND DISCUSSION

In this section, the effect of various structural parameters in relation to the dynamic response of the system have been conducted using the following numerical data, $\mu = 0.075$; $EI = 16000$; $g = 10$; $L = 12$; $\mu_0 = 0.10$, $0.20, 0.30$ and 0.35 ; $x = \frac{L}{2}$; $k = 10$; $\pi = \frac{22}{7}$; $t = 0.5$ and the results obtained were presented through plots. On the basis of the analytical solution, numerical computations were carried out with the aid of Jupyter notebook software, which was programmed in Python. Unless otherwise noted, the material, geometry, and loading data are chosen as the reference values taken from Abu-Hilal (2006) and Li et al., (2016). The dynamics of the model was analysed at the middle point of the beams, $x = L/2$, at the specified time t . The validity of the present formulation was first assessed by comparing the predicted response with the results reported by Abu-Hilal (2006) and Li et al (2016) for a simply supported double Euler-Bernoulli beam system subjected to a moving load. In this comparison, the impact of the nonlinear coupling between variable-stiffness and viscoelastic layers was not considered.

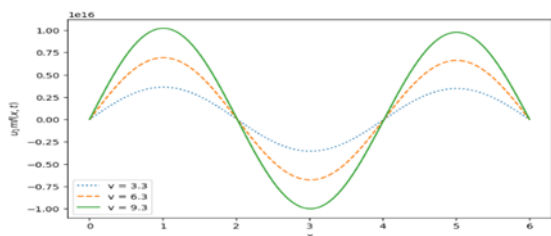


Figure 2: Effect of Variation of Velocity on the Dynamic Response of the (a) Upper Beam Due to Moving Load

Effect of Velocity v

The results in Figure 2 and 3 demonstrate that increasing the velocity values ($v = 3.3\text{ms}^{-1}, 6.3\text{ms}^{-1}, 9.3\text{ms}^{-1}$) of the moving force leads to increase in the response amplitudes of deflection of both beams. For the upper beam, which is directly subjected to the moving force, the increase in amplitude is due to the dynamic excitation introduced as the load travels along the beam at higher speed. The lower beam also exhibits an increase in response amplitude due to the stronger dynamic interaction and load transfer through the nonlinear variable stiffness interconnecting layer. This is in good agreement with the results obtained in Abu-Hilal (2006) where the response amplitudes of deflection increase with the increase in the velocity values. As the moving force velocity increases, the enhanced vibration of the upper beam is transmitted to the lower beam through the coupling mechanism, producing a corresponding increase in its dynamic response despite the restraining effect of the Winkler foundation. This behaviour indicates that the nonlinear variable stiffness layer does not completely suppress the velocity dependent dynamic effects but governs their transmission between the two beams. Hence, the observed increase in response amplitude with moving force velocity highlights the sensitivity of the coupled double beam system to load speed and emphasizes the need to account for velocity effects when assessing the dynamic behaviour and vibration characteristics of structures subjected to moving forces.

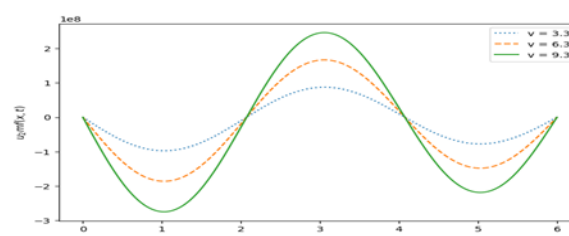


Figure 3: Effect of Variation of Velocity on the Dynamic Response of The Lower Beam Due to Moving Load

Effect of Viscoelastic Parameter ε_0 of the Nonlinear Layer

Figure 4 and 5 show that increasing the viscoelastic parameter ε , ($\varepsilon = 0.1\text{Nsm}^{-1}, 0.2\text{Nsm}^{-1}, 0.3\text{Nsm}^{-1}$) of the interconnecting layer produces a reduction in the response amplitudes of deflection of the upper and lower beams under the moving force. This shows that higher viscoelastic parameter values of the interlayer make the coupling resist the relative movement of the two beams and dissipate the vibration energy produced due to the

action of the applied moving load. The lower beam also experiences reduced response due to the higher energy dissipation in the interlayer, which results in reduced dynamic loading of the lower beam due to the upper beam. Besides, the Winkler foundation acting under the lower beam provides elastic coupling to limit its lateral deflection, thus complementing the dissipation effect of the viscoelastic layer.

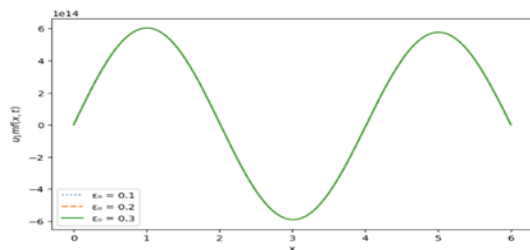


Figure 4: Effect of Viscoelastic Parameter Variation on the Dynamic Response of the Upper Beam Due to Moving Load

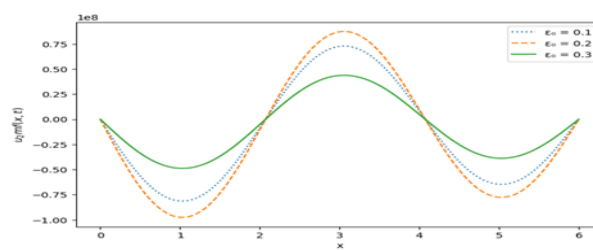


Figure 5: Effect of Viscoelastic Parameter Variation on the Dynamic Response of the Lower Beam Due to Moving Load

Effect of Nonlinear Variable Stiffness Layer k_2

The results in Figure 6 and 7 indicate that increasing the values nonlinear variable stiffness parameter k_2 , ($k_2 = 0.1 \sim 5 \times 10^5 \text{ Nm}^{-2}$) causes the response of the upper beam to approach rigid coupling, while the lower beam experiences reduction in response amplitude of deflection under the moving force. The response near rigid behaviour observed in the upper beam indicates that

an increase in nonlinear stiffness increases the interaction of the two beams in terms of mechanics, making it difficult for the two beams to move apart. As a result, this increases the response of the upper beam in the system as a whole. In other words, there is a distinction in the reaction of the nonlinear variable stiffness in the two structures, one of which is increasing the response of the upper beam in an almost rigid manner, while decreasing the dynamic reaction of the lower beam.

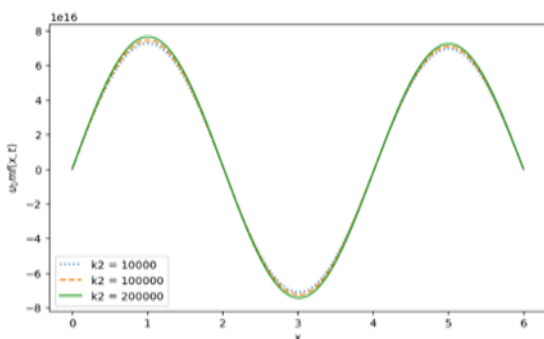


Figure 6: Effect of Nonlinear Stiffness Parameter Variation on the Dynamic Response of the Upper Beam Due to Moving Load

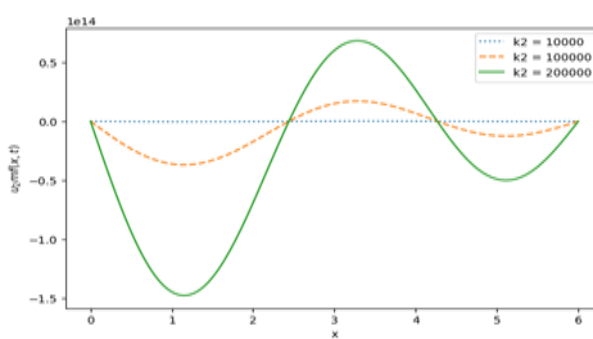


Figure 7: Effect of Nonlinear Stiffness Parameter Variation on the Dynamic Response of the Lower Beam Due to Moving Load

Effect of Stiffness Parameter k_0

In Figure 8 and 9, it is noted that increasing the stiffness parameter k_0 , ($k_0 = 0.01 \sim 5 \times 10^5 \text{ Nm}^{-2}$) of the interconnecting layer increases the upper beam response amplitudes towards a rigid coupling response, whereas the lower beam experiences an increase in response amplitude under the moving force which is in good agreement with those reported in Li et al., (2016). With the increase in stiffness of the interlayer, the response amplitude of deflection of the two beams increases, and

the relative displacement between them becomes small, which makes the upper beam behave more and more like a strongly coupled system. With the higher stiffness values of the interlayer, the response becomes close to the rigid coupling state. Even though the Winkler model gives extra restoring force to the lower beam, the increase in the dynamic load of the lower beam because of the increased stiffness of the interlayer is greater than the restoring force and makes the response of the lower beam increase in the parameter range under study.

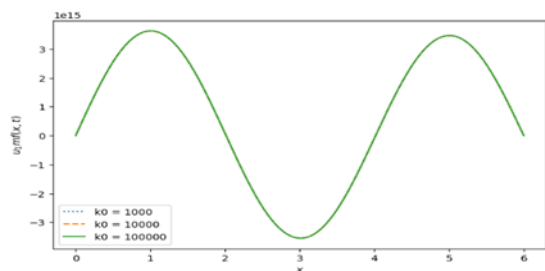


Figure 8: Effect of Stiffness Parameter Variation on the Dynamic Response of the Upper Beam Due to Moving Load

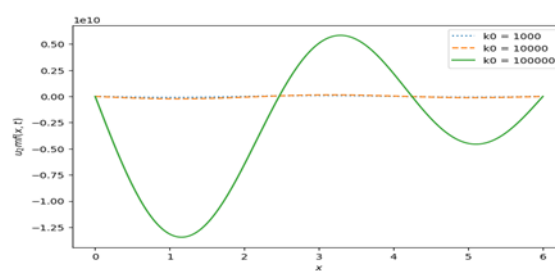


Figure 9: Effect of Stiffness Parameter Variation on the Dynamic Response of the Lower Beam Due to Moving Load

Effect of the Winkler Layer below the Lower Beam k_L

Figure 10 and 11 show the variation of the Winkler foundation parameter k_L , ($k_L = 0.01 \sim 5 \times 10^5 \text{ Nm}^{-2}$) on the upper beam and lower beam. The results show that increasing the Winkler foundation beneath the lower beam causes the response of the upper beam to attain the rigid coupling condition, while the response of the lower beam approaches the rigid coupling response under the moving force. This is in good agreement with the results obtained in Li et al., (2016). This behaviour can be attributed to the higher foundation stiffness, which is responsible for the increased elastic restraining forces

acting on the lower beam to resist the movement of the beam sideways. With the increase in the Winkler parameter, the lower beam will become restrained in its movement from the side due to the action of higher foundation stiffness, making the beam less prone to deformation and hence increasing the interaction between the two beams. This higher restraint from the foundation results in a change in the load distribution mechanism, making the upper beam act more like a rigidly coupled system. With respect to the lower beam, the restraint from the foundation makes the behaviour of the lower beam similar to a rigidly coupled beam because it will deform less.

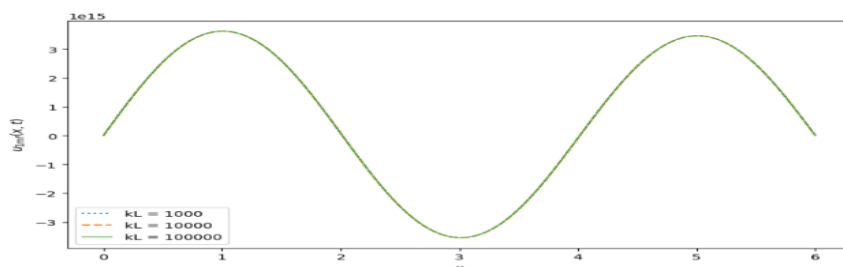


Figure 10: Effect of Stiffness Parameter below the Lower Beam Variation on the Dynamic Response of the Upper Beam Due to Moving Load

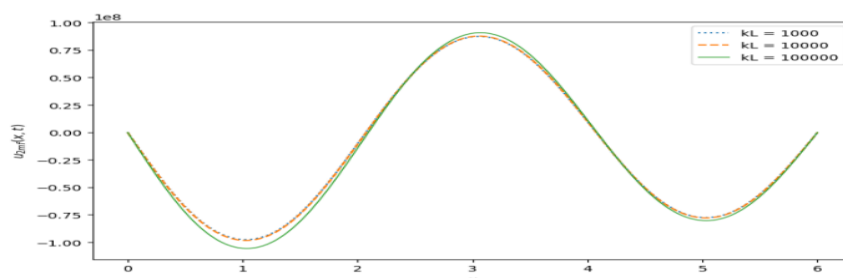


Figure 11: Effect of Stiffness Parameter below the Lower Beam Variation on the Dynamic Response of the Upper Beam Due to Moving Load

CONCLUSION

In this research, the dynamics of the nonlinear double beam with varying stiffness and viscoelasticity was considered under a moving load, in which a semi-analytical solution for the dynamic response of the upper

and lower beams was obtained by combining the finite Fourier sine integral transform with the differential transform method. The numerical simulations reveal that the increase in the velocity of the moving force leads to the increase in the response of the beams. Due to

increased energy dissipation, an increase in the viscoelastic parameter decreases the response. Moreover, an increase in the nonlinear variable stiffness parameter and the interlayer stiffness increase the coupling and hence drive the upper beam closer to the rigid coupling response, but increase the response of the lower beam. Due to the higher elastic restraint, the increase in the Winkler foundation parameter drives both beams towards the rigid coupling. The findings thus show that the behaviour of the viscoelastic layer and nonlinear variable stiffness significantly control the vibration and coupling of the beams. A viscoelastic interlayer is thus beneficial for isolating vibrations on railway tracks and layered decks, while a stiff interlayer improves load sharing between the beams.

Nevertheless, it should be noted that the above conclusions are based on the Euler-Bernoulli model, where shear deformations and rotatory inertias have been disregarded. Therefore, in some cases, the use of the Euler-Bernoulli beam theory may lead to errors when modelling thick beams and higher mode vibrations. Moreover, the perfect interlayer model and an approximate solution via the Differential Transformation Method, depending on convergence and truncation, may not reflect true system characteristics. In spite of this, the above conclusions provide valuable insights into the design of layered structures.

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