

Vibrational Behaviour of a Double Rayleigh Beam System coupled by Damped Pasternak Layer under a Moving Load



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ABSTRACT

This study analysed the dynamic response of a Pasternak damped double Rayleigh beam system subjected to a concentrated moving load. The two parallel beams are interconnected by a viscoelastic Pasternak layer, with the moving load applied to the upper beam. The study incorporate rotary inertia, interlayer stiffness, shear interaction, damping, and moving load velocity. A finite Fourier sine integral transform is deployed initially to reduce the governing partial differential equations while differential transform methods (DTM) are subsequently employed to obtain approximate analytical dynamic responses. Parametric analyses are conducted to examine the effects of moving load velocity, viscoelasticity, rotary inertia, interlayer stiffness, and Pasternak shear parameter on the transverse responses of the beams. The results show that the velocity of the moving load causes deflection in the entire system, while the rotary inertia tends to reduce these deflections. Viscoelasticity reduces the vibration of the lower beam, causing the upper beam to behave like a rigid body due to better energy absorption. However, stiffening the interlayer and Pasternak shear increases the deflection of the upper beam but increases the vibration of the lower beam due to increased load transfer. The findings clearly demonstrated the importance of rotatory inertia and combined effect of viscoelastic and shear coupling in dynamic load distribution of a double Rayleigh beam system.

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INTRODUCTION

Dynamic applications of moving loads on structural element such as beams are widely used in transportation engineering particularly in railways, road and highway bridges (Stojanovic & Kozic 2015). Beams may be elastic or inelastic, but this particular study examines elastic beams, which are the most common and important types of structural members. The theoretical treatment of elastic structures on elastic layer plays a significant role in modern engineering structures and mechanics with a broad application in the analysis and design of building foundations, railway track beds, highway pavements, and various other geotechnical infrastructures. The dynamic stability and oscillatory response of such systems have attracted sustained scholarly and engineering interest for many years (Michaltsos *et al.*, 1996; Zhang & Zheng 2009; Ma *et al.*, 2024). In fact, it is a fundamental aspect of structural and geotechnical engineering research. As a matter of fact, various beam models have been examined by scholars and among them are the Euler–Bernoulli,

Timoshenko, Rayleigh, and shear deformable theories in formulating the mathematical models of beam vibration (Pasternak, 1954; Abu-Hilal, 2006; Mohammadi & Nasirshoabi, 2015; Hamed *et al.*, 2020; Adali, 2023). Elastic structural systems can be classified as single beam, double beam, or multiple beam systems. However, this particular study focuses on double beam systems. In comparison with single beam models, the study of vibration in double beam models presents greater analytical and computational challenges owing to the interaction between the coupled beams. Consequently, research in this area remains relatively limited despite its practical significance (Michaltsos *et al.*, 1996; Abu-Hilal 2006; Li *et al.*, 2016; Usman *et al.*, 2018; Jiang *et al.*, 2019).

A double beam model is a type of structural configuration consisting of two parallel beams joined together by a spring type or viscoelastic interlayer which may be of the Winkler model, the Pasternak model, or a hybrid of the two. A comprehensive review of various beam models is

presented in the text (Stojanovic & Kozic, 2015), which discusses the vibration behaviour of Euler–Bernoulli, Rayleigh, Timoshenko, and Reddy–Bickford beam configurations. The vibration characteristics of coupled Euler–Bernoulli and Rayleigh beam pairs have been extensively investigated by various researchers, who have employed analytical, numerical, and approximate techniques to obtain solutions for both free and forced vibration problems (Abu-Hilal, 2006; Gbadeyan & Hammed, 2017; Usman *et al.*, 2023; Hang *et al.*, 2024). System of double beams were developed to address some of the limitations of single beam systems, some of these shortcomings include low load carrying and limited resistance to bending, torsion, and lateral loads (Gbadeyan & Hammed, 2017; Koziol & Pilecki, 2018; Hammed *et al.* 2025; Lu *et al.*, 2026). These systems offer increased load bearing capacity and resistance to various types of loads. Moreover, double beam configurations can be engineered to achieve larger spans when compared to single beams, rendering them appropriate for use in larger scale structures. Thus, this study examined the effect of moving load on the dynamic response of a Pasternak double Rayleigh beam system. Despite these studies, the existing models have certain mathematical and physical drawbacks. In many cases, Euler–Bernoulli beams are used, where rotary inertia is ignored (Abu-Hilal, 2006; Li *et al.*, 2016; Koziol & Pilecki 2018; Hammed *et al.*, 2020), whereas for the

inter-layer, Winkler layer is used without taking into account any shear interaction, although Pasternak layer can be used but in some cases, damping is not taken into account (Mohammadi & Nasirshoabi, 2015; Hammed *et al.*, 2025). The present study addresses these limitations by incorporating rotary inertia, damping, and Pasternak shear interaction in a single coupled model.

This study aims to investigate the dynamic response of a double Rayleigh beam system interconnected by a damped Pasternak layer and subjected to a moving concentrated load, with emphasis on the effects of velocity, rotary inertia, viscoelastic damping, interlayer stiffness, and shear interaction.

MATERIALS AND METHODS

The physical model of the dynamic response of a Pasternak damped double Rayleigh beam subjected to a moving concentrated load as shown in Figure 1 is considered. The structural model consist of two finite, uniform, undamped, parallel upper and lower Rayleigh beams of the same length L , and mass per unit length μ . The two beams are interconnected through a damped Pasternak type layer, which is represented mechanically by parallel springs and dashpots together with a shear layer. The structural model is considered within a single plane, and axial deformation is neglected.

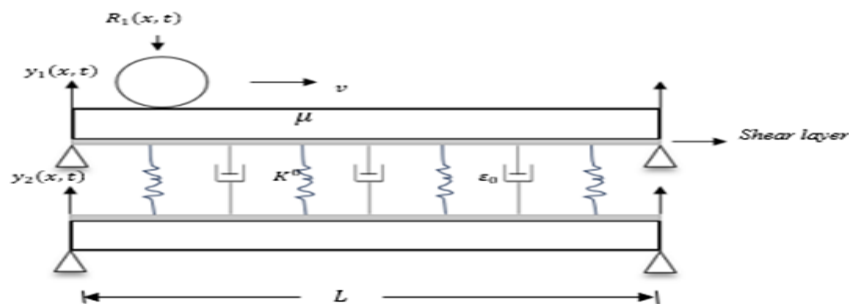


Figure 1: A Double Rayleigh Beam System Subjected to a Moving Load. (Abu-Hilal 2006; Mohammadi & Nasirshoabi 2015)

The transverse vibrational response of the upper and lower Rayleigh beams is expressed as $y_1(x, t)$ and $y_2(x, t)$ respectively, and are described by the following coupled set of fourth order differential equations of motion (Abu-Hilal, 2006; Mohammadi & Nasirshoabi, 2015; Gbadeyan & Hammed, 2017).

$$e_1 r_1 \frac{\partial^4 y_1(x, t)}{\partial x^4} + \mu \frac{\partial^2 y_1(x, t)}{\partial t^2} - T_{c1} \mu R \frac{\partial^4 y_1(x, t)}{\partial t^2 \partial x^2} + T_{c2} \left[\left(K^0 + \epsilon_0 \frac{\partial}{\partial t} \right) (y_1(x, t) - y_2(x, t)) + G^0 \left(\frac{\partial^2 y_1(x, t)}{\partial x^2} - \frac{\partial^2 y_2(x, t)}{\partial x^2} \right) \right] = R_1(x, t) \quad (1)$$

$$e_2 r_2 \frac{\partial^4 y_2(x, t)}{\partial x^4} + \mu \frac{\partial^2 y_2(x, t)}{\partial t^2} - T_{c1} \mu R \frac{\partial^4 y_2(x, t)}{\partial t^2 \partial x^2} + T_{c2} \left[\left(K^0 + \epsilon_0 \frac{\partial}{\partial t} \right) (y_2(x, t) - y_1(x, t)) + G^0 \left(\frac{\partial^2 y_2(x, t)}{\partial x^2} - \frac{\partial^2 y_1(x, t)}{\partial x^2} \right) \right] = 0 \quad (2)$$

where e_i is the Young's modulus, r_i is the second moment of inertia, R is the rotatory inertia correction factor, G^0 is the shear coefficient, ϵ_0 is the damped coefficient, K^0 is the stiffness constant, t is the time, T_{c1} and T_{c2} are tracing constants and $R_1(x, t)$ is the dynamic load function described as

$$R_1(x, t) = r_0 \delta(x - vt) \quad (3)$$

Where r_0 is the constant force traversing the length of the beam and $\delta(x - vt)$ is the Dirac delta function defined as,

$$\delta(x - vt) = \frac{1}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \cos \frac{n\pi vt}{L} \cos \frac{n\pi x}{L} \quad (4)$$

For the problem under study, the boundary conditions for Equations (1) and (2) are chosen as,

$$y_1(0, t) = 0 = y_1(L, t) \quad (5)$$

$$\frac{\partial^2 y_1(0, t)}{\partial x^2} = 0 = \frac{\partial^2 y_1(L, t)}{\partial x^2} \quad (6)$$

$$y_2(0, t) = 0 = y_2(L, t) \quad (7)$$

$$\frac{\partial^2 y_2(0, t)}{\partial x^2} = 0 = \frac{\partial^2 y_2(L, t)}{\partial x^2} \quad (8)$$

And the corresponding initial conditions are:

$$y_1(x, 0) = 0 = \frac{\partial y_1(x, 0)}{\partial t} \quad (9)$$

$$y_2(x, 0) = 0 = \frac{\partial y_2(x, 0)}{\partial t} \quad (10)$$

Finite Fourier Sine Integral Transform

In this section, the unknown transverse displacement functions $y_1(x, t)$ of the upper beam and $y_2(x, t)$ lower beam were sought by implementing the generalized finite Fourier sine integral transformation to Equations (1) and (2). The integral transform deployed is,

$$\bar{y}_m(n, t) = \int_0^L y_m(x, t) U_n(x) dx; m = 1, 2; n = 1, 2, 3, \dots \quad (11)$$

The corresponding inverse transform of Equations (11) is $y_m(x, t) = \frac{2}{L} \sum_{n=1}^{\infty} \bar{y}_m(n, t) U_n(x), m = 1, 2$ (12)

Where

$$U_n(x) = \sin \frac{\lambda_n x}{L} \quad \lambda_n = n\pi \quad (13)$$

Are assumed for the upper and lower beams respectively. Then, the unknown time functions, $\bar{y}_1(n, t)$ and $\bar{y}_2(n, t)$ are the modal amplitudes to be determined later, while $U_n(x), n = 1, 2, 3, \dots$ are the known normalized mode shape functions corresponding to the n th vibration mode of the beam.

Applying Equations (11) - (13) into Equations (1) and (2), the resulting equations are,

$$\begin{aligned} & \ddot{\bar{y}}_1(n, t) + \omega_n^2 \bar{y}_1(n, t) + T_{c1} R b_1^*(t) \\ & + \frac{T_{c2}}{\mu} [b[\bar{y}_1(n, t) - \bar{y}_2(n, t)] + \varepsilon_0 [\dot{\bar{y}}_1(n, t) - \\ & \dot{\bar{y}}_2(n, t)]] = \frac{r_0}{\mu L} U_n(vt) \end{aligned} \quad (14)$$

$$\begin{aligned} & \ddot{\bar{y}}_2(n, t) + \omega_n^2 \bar{y}_2(n, t) + T_{c1} R b_1^*(t) + \frac{T_{c2}}{\mu} [b[\bar{y}_2(n, t) - \bar{y}_1(n, t)] + \\ & \varepsilon_0 [\dot{\bar{y}}_2(n, t) - \dot{\bar{y}}_1(n, t)]] = 0 \end{aligned} \quad (15)$$

Where

$$b_1(t) = \frac{n^2 \pi^2}{L^2} \bar{y}_1(n, t), \quad b_2^*(t) = \frac{n^2 \pi^2}{L^2} \ddot{\bar{y}}_2(n, t) \quad (16)$$

$$b = K^0 - G^0 \frac{n^2 \pi^2}{L^2}, \quad \omega_n^2 = e_2 T_2 \frac{n^4 \pi^4}{\mu L^4} = e_1 r_1 \frac{n^4 \pi^4}{\mu L^4} \quad (17)$$

ω_n is the natural circular frequency.

In this study, $e_1 r_1 = e_2 r_2$, implying which implies that both beams possess identical bending stiffness, resulting in symmetric vibration behaviour, uniform load distribution, and stronger coupled motion.

To obtain inherent modified frequency I of the rotatory inertia which is associated with the reduced equations,

T_{c2} , is set equal to zero in Equations (14) and (15) to have,

$$\ddot{\bar{y}}_1(n, t) + \omega_n^2 \bar{y}_1(n, t) + T_{c1} R \frac{n^2 \pi^2}{L^2} \bar{y}_1(n, t) = \frac{r_0}{\mu L} U_n(vt) \quad (18)$$

$$\ddot{\bar{y}}_2(n, t) + \omega_n^2 \bar{y}_2(n, t) + T_{c1} R \frac{n^2 \pi^2}{L^2} \bar{y}_2(n, t) = 0 \quad (19)$$

The operation conducted which produced the resulting equations (18) and (19) ensures the disconnected equations and ease of analysis of the coupled Equations (14) and (15).

Further simplification of Equations (18) and (19) yields,

$$(1 + T_{c1} R \frac{n^2 \pi^2}{L^2}) \ddot{\bar{y}}_1(n, t) + \omega_n^2 \bar{y}_1(n, t) = \frac{r_0}{\mu L} U_n(vt) \quad (20)$$

$$(1 + T_{c1} R \frac{n^2 \pi^2}{L^2}) \ddot{\bar{y}}_2(n, t) + \omega_n^2 \bar{y}_2(n, t) = 0 \quad (21)$$

Hence, Equations (20) and (21) becomes

$$\ddot{\bar{y}}_1(n, t) + \omega_{ns}^2 \bar{y}_1(n, t) = \theta^* U_n(vt) \quad (22)$$

$$\ddot{\bar{y}}_2(n, t) + \omega_{ns}^2 \bar{y}_2(n, t) = 0 \quad (23)$$

Where

$$\omega_{ns}^2 = \frac{\omega_n^2}{1 + T_{c1} R \frac{n^2 \pi^2}{L^2}} \quad (24)$$

$$\theta^* = \frac{\frac{r_0}{\mu L}}{1 + T_{c1} R \frac{n^2 \pi^2}{L^2}} \quad (25)$$

The following equations were obtained by $T_{c2} = 1$ in Equations (22) and (23)

$$\begin{aligned} & \ddot{\bar{y}}_1(n, t) + \omega_{ns}^2 \bar{y}_1(n, t) + \frac{1}{\mu} [b[\bar{y}_1(k) - \bar{y}_2(k)] + \\ & \varepsilon_0 [(k+1)\bar{y}_1(k+1) - (k+1)\bar{y}_2(k+1)]] = \\ & Z_1 L U_n(vt) \end{aligned} \quad (26)$$

$$\begin{aligned} & \ddot{\bar{y}}_2(n, t) + \omega_{ns}^2 \bar{y}_2(n, t) + \frac{1}{\mu} [b[\bar{y}_2(k) - \bar{y}_1(k)] + \\ & \varepsilon_0 [(k+1)\bar{y}_2(k+1) - (k+1)\bar{y}_1(k+1)]] = 0 \end{aligned} \quad (27)$$

Where,

$$Z_1 = \frac{\frac{M_L}{\mu}}{1 + T_{c1} R \frac{n^2 \pi^2}{L^2}} \quad (28)$$

$$N = \frac{1}{\mu} \quad (29)$$

Differential Transformation Method

This section discusses some basic idea of differential transform method by taking into account an analytical function $\bar{y}_m(n, t)$ with continuous derivative within a sufficiently large domain Thus,

$$y_m(k) = \frac{1}{k!} \left[\frac{d^k \bar{y}_m(n, t)}{dt^k} \right]_{t=t_0} \quad (30)$$

Where $\bar{y}_m(n, t)$ denotes the original function, while $y_m(k)$ represents the corresponding transformed function. The inverse transformation of $y_m(k)$ as represented in Hosseinzadeh and Salehpour (2013) is measured in:

$$\bar{y}_m(n, t) = \sum_{k=0}^{\infty} y_m(k) (t - t_0)^k \quad (31)$$

Considering Equations (30) and (31), the resulting equation is,

$$\bar{y}_m(n, t) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} \frac{d^k \bar{y}_m(n, t)}{dt^k} \Big|_{t=t_0} \quad (32)$$

When t_0 is set equal to 0, in Equation (31), the equation obtained is,

$$\bar{y}_m(n, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left. \frac{d^k \bar{y}_m(n, t)}{dt^k} \right|_{t=0} \quad (33)$$

Hence,

$$\bar{y}_m(n, t) = \sum_{k=0}^{\infty} y_m(k) t^k \quad (34)$$

For practical applications such as the present study, the function $\bar{y}_m(n, t)$ can be approximated by finite series, allowing Equation (31) to be written as follows

$$\bar{y}_m(n, t) = \sum_{k=0}^N y_m(k) t^k, \quad m = 1, 2 \quad (35)$$

Therefore, the term $\sum_{k=N+1}^{\infty} y_m(k) t^k$ is assumed to be insignificant, and the value of N is derived based on the convergence of vibrational frequency as employed in Hosseinzadeh and Salehpour (2013) and used in this study in Table 1.

Table 1: Basic Theorems on Differential Transform Method for Equations of Motion

Original Function	Transformed Function
$y(t) = u(t) + v(t)$	$Y(k) = U(k) + V(k)$
$y(t) = cu(t)$	$Y(k) = cU(k)$
$y(t) = \frac{du(t)}{dt}$	$Y(k) = (k+1)\bar{U}(k+1)$
$y(t) = \frac{d^2u(t)}{dt^2}$	$Y(k) = (k+1)(k+2) \dots (k+n-1)\bar{U}(k+n)$
$y(t) = u(t)v(t)$	$Y(k) = \sum_{k=0}^{\infty} \bar{U}(n)\bar{V}(k-n)$
$y(t) = u(t) \int_0^L v(x)dx$	$Y(k) = \sum_{k=0}^{\infty} \bar{U}(k-n) \frac{\bar{V}(n-1)}{n} \quad k \leq 1$
$y(t) = \sin at$	$Y(k) = \frac{1}{k!} a^k \sin\left(\frac{k\pi}{2}\right)$
$y(t) = \cos at$	$Y(k) = \frac{1}{k!} a^k \cos\left(\frac{k\pi}{2}\right)$
$y(t) = \sinh at$	$Y(k) = \frac{1}{2k!} [a^k - (-a)^k]$
$y(t) = \cosh at$	$Y(k) = \frac{1}{2k!} [a^k + (-a)^k]$

Table 1 provides a summary of some operational rules for the conversion of the dynamic equations and the boundary conditions.

The concept of inverse differential transform and inverse differential transform, as discussed earlier, is now applied to analyse and solve Equations (26) and (27) to obtain the relation below

$$\bar{y}_1(k+2) = \frac{1}{(k+2)(k+1)} \left[\frac{Z_1}{k!} \left(\frac{n\pi v}{L} \right) \sin\left(\frac{k\pi}{2}\right) - \omega_{ns}^2 \bar{y}_1(k) - N[b\bar{y}_1(k) - \bar{y}_2(k)] + \varepsilon_0[(k+1)\bar{y}_1(k+1) - (k+1)\bar{y}_2(k+1)] \right] \quad (36)$$

$$\bar{y}_2(k+2) = -\frac{1}{(k+2)(k+1)} \left[\omega_{ns}^2 \bar{y}_2(k) + N[b\bar{y}_2(k) - \bar{y}_1(k)] + \varepsilon_0[(k+1)\bar{y}_2(k+1) - (k+1)\bar{y}_1(k+1)] \right] \quad (37)$$

The transformed initial conditions using Table 1:

$$\bar{y}_1(0) = 0 = \bar{y}_1(1) \quad (38)$$

$$\bar{y}_2(0) = 0 = \bar{y}_2(1) \quad (39)$$

Thus, for $k = 0, 1, 2, 3, \dots$, in Equations (36) and (37), the following solution were derived:

For: $k = 0$;

$$\bar{y}_1(2) = \frac{1}{2}(0) = 0 \quad (40)$$

$$\bar{y}_2(2) = \frac{1}{2}(0) = 0 \quad (41)$$

$k = 1$:

$$\bar{y}_1(3) = \frac{Z_1}{3!} \frac{n\pi v}{L} \quad (42)$$

$$\bar{y}_2(3) = 0. \quad (43)$$

$k = 2$;

$$\bar{y}_1(4) = -\frac{Z_1}{4!} \frac{N\varepsilon_0 n\pi v}{L} \quad (44)$$

$$\bar{y}_2(4) = \frac{Z_1}{4!} \frac{N\varepsilon_0 n\pi v}{L} \quad (45)$$

$k = 3$;

$$\bar{y}_1(5) = -\frac{Z_1}{5!} \frac{n\pi v}{L} \left[\left(\frac{n\pi v}{L} \right)^2 + \omega_{ns}^2 + N[b - 2N\varepsilon_0^2] \right] \quad (46)$$

$$\bar{y}_2(5) = \frac{Z_1}{5!} \frac{n\pi v}{L} N[b - 2N\varepsilon_0^2] \quad (47)$$

$k = 4$;

$$\bar{y}_1(6) = \frac{Z_1 N\varepsilon_0}{6!} \frac{n\pi v}{L} \left[\left(\omega_{ns}^2 - \left(\frac{n\pi v}{L} \right)^2 \right) - \omega_{ns}^2 + 4N^2 \varepsilon_0^2 \right] \quad (48)$$

$$\bar{y}_2(6) = \frac{Z_1 N\varepsilon_0}{6!} \frac{n\pi v}{L} \left[-\omega_{ns}^2 - 1 \left(\left(\frac{n\pi v}{L} \right)^2 - \omega_{ns}^2 + 4N[b - N\varepsilon_0^2] \right) \right] \quad (49)$$

Applying Equations (38)-(49) in the inverse differential transformation of Equation (12) yields, respectively.

$$y_1(x, t) = \sum_{n=1}^2 \left[\frac{Z_1}{3!} \left(\frac{n\pi v}{L} \right) t^3 - \frac{Z_1}{4!} \left(\frac{n\pi v}{L} \right) N\varepsilon_0 t^4 - \frac{Z_1}{5!} \left(\frac{n\pi v}{L} \right) \left[\left(\frac{n\pi v}{L} \right)^2 + \omega_{ns}^2 + N[b - 2N\varepsilon_0^2] \right] t^5 + \frac{Z_1 N\varepsilon_0}{6!} \left(\frac{n\pi v}{L} \right) \left[\left(\omega_{ns}^2 - \left(\frac{n\pi v}{L} \right)^2 \right) - \omega_{ns}^2 + 4N^2 \varepsilon_0^2 \right] t^6 + \dots \right] \sin \frac{n\pi x}{L}, \quad (50)$$

$$y_2(x, t) = \sum_{n=1}^2 \left[\frac{Z_1}{4!} N\varepsilon_0 \left(\frac{n\pi v}{L} \right) t^4 + \frac{Z_1}{5!} \left(\frac{n\pi v}{L} \right) N[b - 2N\varepsilon_0^2] t^5 + \frac{Z_1 N\varepsilon_0}{6!} \left(\frac{n\pi v}{L} \right) \left[-\omega_{ns}^2 - 1 \left(\left(\frac{n\pi v}{L} \right)^2 - \omega_{ns}^2 + 4N[b - N\varepsilon_0^2] \right) \right] t^6 + \dots \right] \sin \frac{n\pi x}{L} \quad (51)$$

Thus, Equation (50) and (51) are the dynamic response equation of deflection of the simply supported upper and lower Rayleigh beams traversed by a concentrated moving force.

RESULTS AND DISCUSSION

In this section, numerical experimentation was conducted by introducing a suitable programming software called Python Jupyter note. The results obtained have been presented in plots for the ease of interpretation. In order to achieve this, the dynamic responses $y_i(x, t)$, $i = 1, 2$ against the position of the load ($x = vt$) at some specified time (t) for both the simply supported upper and lower Rayleigh beams. The material, geometric and loading parameters were assigned the reference values reported

in Abu-Hilal (2006) and Gbadeyan and Hammed (2017). The study examined the impact of the combined and individual impacts of velocity of the moving load v , ($v = 3.3, 6.3, 9.3$), stiffness parameter k^0 ($k^0 = 1 \sim 5 \times 10^5 \text{ N/m}^2$), shear parameter G_0 , ($G_0 = 10, 100, 1000$), rotatory inertia effect R_0 ($R_0 = 4, 6, 8$) and the viscoelastic parameter ε_0 ($\varepsilon_0 = 0.10, 0.20, 0.25, 0.35 \text{ Ns/m}^2$) on the upper and lower beams response due to moving load using the following numerical data: $\mu = 0.075 \text{ kg/m}$; $EI = 16000 \text{ Nm}^2$; $g = 10 \text{ m/s}^2$; $L = 6 \text{ m}$; $k = 1 \sim 5 \times 10^5 \text{ N/m}^2$; $\pi = \frac{22}{7}$; $t = 0.5$

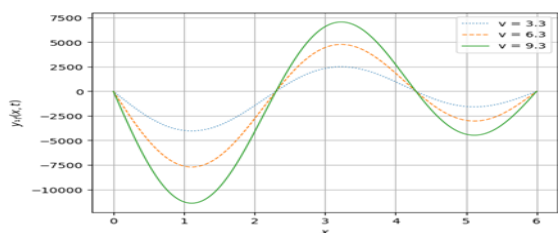


Figure 2: Effect of Velocity On the Transverse Deflection of the Upper Beam Due to a Moving Force

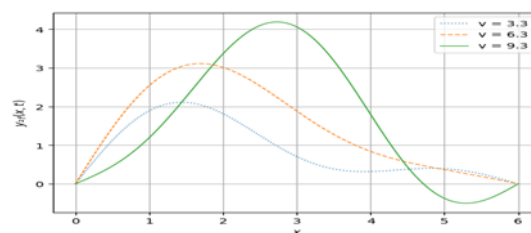


Figure 3: Effect of Velocity on the Transverse Deflection of the Lower Beam Due to Moving Force

Effect of Velocity on the Transverse Deflection of the Pasternak Damped Double Rayleigh Beam

The results in Figures 2 and 3 show that an increase in the velocity of the moving force increases the transverse deflection of both the upper and lower Rayleigh beams. Increasing the velocity value from $v = 3.3 \text{ ms}^{-1}$ to $v = 9.3 \text{ ms}^{-1}$ increases the maximum deflection of the upper beam and that of the lower beam. The increase is primarily associated with increased dynamic excitation and rotatory effects due to higher travelling speeds,

which bring the frequency of the load closer to the natural frequencies of the system and so raise the dynamic amplification. The increase is primarily associated with increased dynamic excitation and rotatory effects due to higher travelling speeds, while the response of the lower beam is characterised by dynamic force transmission through the Pasternak interlayer. Hence, neglecting the velocity dependent nature of the moving excitation may lead to an underestimation of the actual deformation experienced by coupled beam systems subjected to moving loads at high speed

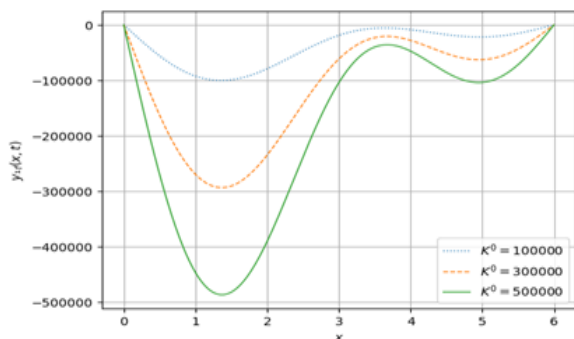


Figure 4: Effect of Stiffness Parameter (k^0) on the Transverse Deflection of the Upper Beam under a Moving Force

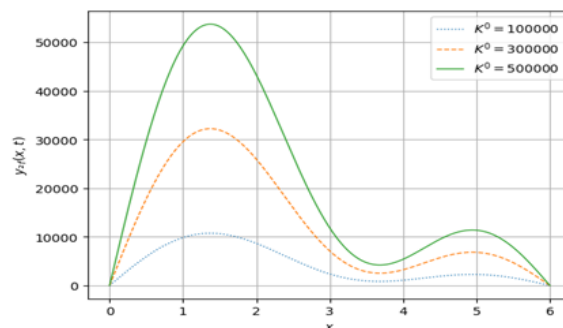


Figure 5: Effect of Stiffness Parameter (k^0) on the Transverse Deflection of the Lower Beam Under a Moving Force

Effect of Stiffness Parameter (k^0) on the Transverse Deflection of the Pasternak Damped Double Rayleigh Beam

The increase in stiffness parameter values ($k^0 = 1 \sim 5 \times 10^5 \text{ N/m}^2$) in Figure 4 produces a stiffening effect on the upper beam, thereby reducing its deflection amplitude, while simultaneously enhancing the transfer of dynamic excitation to the lower beam in Figure 5 and increasing its response amplitude. Thus, the stiffness parameter serves not only as a deformation control parameter but also as a key mechanism governing load transfer and interaction between the upper and lower beams. As the stiffness parameter increases, the relative

displacement between the two beams decreases, so the two responses move closer to each other and towards the rigid coupling limit, in which both beams deflect together. This is in good agreement with the results obtained in Abu-Hilal (2006) and Gbdeyan & Hammed (2017). The upper beam therefore passes part of the moving-force excitation to the lower beam, which explains why the two beams respond in opposite directions. The result is relevant to coupled beam systems in which increasing the value of the stiffness layer parameter improve the resistance of the upper beam increasing the dynamic demand imposed on the lower beam.

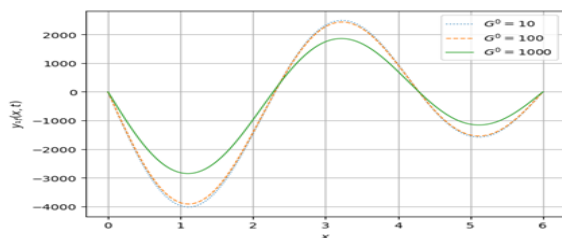


Figure 6: Effect of Shear on the Transverse Deflection of the Upper Beam Under a Moving Force

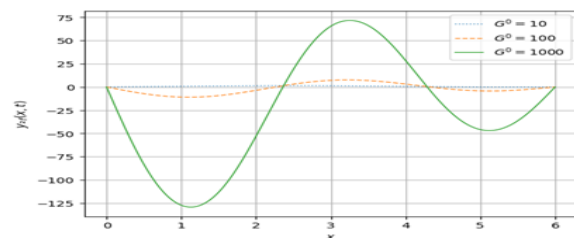


Figure 7: Effect of Shear on the Transverse Deflection of the Lower Beam Under a Moving Force

Effect of Shear Parameter on the Transverse Deflection of the Pasternak Damped Double Rayleigh Beam

In Figure 6 and 7, the dynamic deflection of both the upper and lower Rayleigh beams under a moving force is considered for various values of shear parameter ($G_0 = 10, 100, 1000$). Increasing the values of shear parameter of the Pasternak interlayer, decreases the deflection amplitudes of the upper beam but increases that of the lower beam. This decrease in the response of the upper beam is attributed to the increase in the shear stiffness of the layer between the two beams, thus the resistance to lateral bending is higher, making the beam less flexible. This is in line with prior research on Rayleigh beams on Pasternak foundation (Mohammadi & Nasirshoabi, 2015), where increasing the shear modulus was found to suppress the transverse response. However, the increased response amplitudes of the lower beam is caused by

stronger dynamic force transfer through the layer between the beams. As the shear parameter increases, the relative displacement between the two beams is increasingly restricted, causing stronger coupling and transferring a greater portion of the moving force induced excitation from the upper to the lower beam. Thus, the variation in responses indicate a redistribution of the dynamic deformation within the coupled system. In this case, the upper beam becomes more restrained, while the lower beam becomes more dynamically engaged. The observed behaviour demonstrates that the Pasternak shear does not only provides additional resistance to deformation but also plays an important role in controlling load transmission and interaction between the two Rayleigh beams.

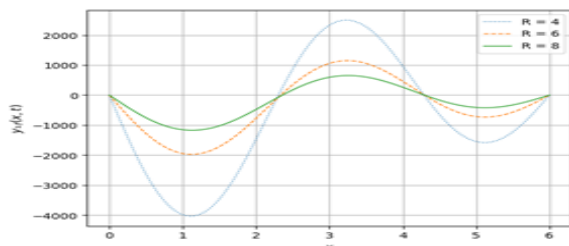


Figure 8: Effect of Rotatory Inertia R on the Transverse Deflection of the Upper Beam under a Moving Force

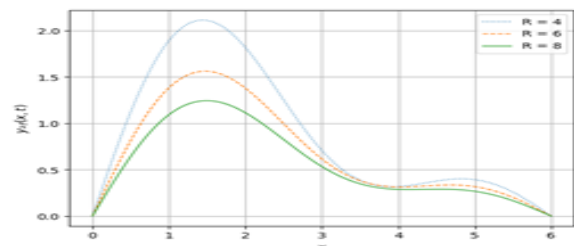


Figure 9: Effect of Rotatory Inertia R on the Transverse Deflection of the Lower Beam under a Moving Force

Effect of Rotatory Inertia on the Transverse Deflection of the Pasternak Damped Double Rayleigh Beam

Figure 8 and 9 shows the variation of rotatory inertia correction factor ($R_0 = 4, 6, 8$) on the dynamic responses of both the upper and lower Rayleigh beams excited by moving force. The results show that increasing the rotatory inertia correction factor reduces the deflection amplitudes of the two beams. This reduction is due to the increased resistance to rotation because of the rotatory inertia and the decrease in dynamic force transfer between the two beams. Thus, the rotatory inertia is one

of the crucial parameters that affect the response of the double Rayleigh beams resting on the Pasternak foundation system subjected to moving loads. This study observation is also supported by some recent studies on Rayleigh beams on Pasternak foundations (Mohammadi & Nasirshoaibi, 2015; Gbadeyan & Hammed, 2017; Usman *et al.*, 2018), where rotary inertia correction factor has been identified as an important parameter in reducing dynamic response.

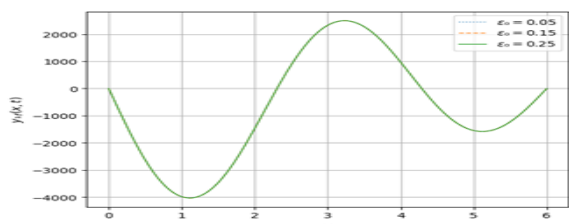


Figure 10: Effect of Viscoelastic on the Transverse Deflection of the Upper Beam under a Moving Force

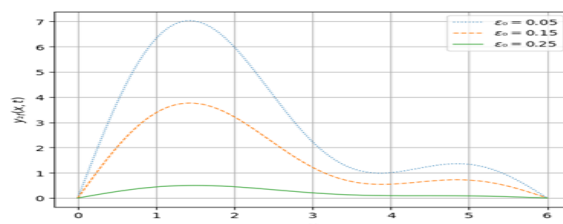


Figure 11: Effect of Viscoelastic on the Transverse Deflection of the Lower Beam under a Moving Force

Effect of Viscoelastic Parameter on the Transverse Deflection of the Pasternak Damped Double Rayleigh Beam

The dynamic effect of viscoelastic parameter value ($\varepsilon_0 = 0.10, 0.20, 0.25, 0.35 \text{Ns/m}^2$) on the response amplitude of deflection of the upper and lower Rayleigh beam is shown in Figure (10) and (11). From the Figure, it can be seen that increasing the values of viscoelastic parameter leads to increasing the coupling between the two beams where very high values of these dynamic parameter lead to rigid coupling of the beams and thereby reducing the response amplitudes of the lower beam. The increase in viscoelasticity increases the resistance to relative movement of the interconnecting layer and causes more of the vibration energy released by the moving force to be dissipated. As a consequence, the upper beam becomes more and more restricted by the interacting structure and starts responding like a rigidly coupled structure. On the other hand, the higher energy dissipation within the viscoelastic layer leads to a reduction in the excitation energy transferred to the lower beam and thus results in reduced deflections of this beam. This interpretation is consistent with earlier studies (Abu-Hilal, 2006; Li *et al.*, 2016; Gbadeyan & Hammed, 2017), showing that the viscosity of a Pasternak foundation has a significant influence on beam deflection under moving loads. Thus, the observed behaviour demonstrates the dual role of the viscoelastic layer includes, strengthening the interaction between the beams while reducing the transmitted vibration.

The deflections of both the upper and lower beams are obtained after due analysis of the beam system. The effects of various dynamic parameters such as velocity of the moving load, viscoelastic parameter, rotatory inertia correction factor, stiffness of the layered parameter, and Pasternak shear layer parameter were examined. From the exploration, it is found that increasing the velocity of the moving load increases the response amplitude of deflection of the beams, thereby validating the importance of velocity dependence effect in the dynamic behaviour of the present coupled beam system. The result reveals that increasing the viscoelastic parameter value of the layer between the two beams reduces response amplitudes of deflection of both beams. However, very high values of viscoelastic parameter lead to rigid coupling of the two beams, which thus vibrate together as a single unit. Rotary inertia correction factor exhibits the same stabilizing influence, as higher values reduce the response of both beams by increasing resistance to rotational acceleration and dynamic deformation.

Conversely, increasing the interlayer stiffness and Pasternak shear parameter reduce the deflection of the beam with which the load is directly in contact while increasing the response amplitudes of the lower beam. This divergence in behaviour stems from stronger elastic and shear coupling, which limits relative motion between the beams while promoting greater transmission and redistribution of the moving-load-induced excitation toward the lower beam. Accordingly, the stiffness and shear parameters function primarily as regulators of dynamic load transfer and deformation distribution, rather than as agents of uniform vibration suppression.

CONCLUSION

However, these results are valid only for the considered case of the assumed double Rayleigh beam model with simply supported ends, damped Pasternak interlayer and moving load on the beam. This is because the approximate solution is based on the finite Fourier sine transform and differential transform technique. The applicability of Rayleigh beam theory does not consider the effect of shear deformation, which means that the results cannot be generalized to other beam theories and other types of loadings.

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