

A Class of Hybrid Linear Multistep Methods for Solving Initial Value Problems of Second Order Ordinary Differential Equations

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ABSTRACT

In this paper, two - step hybrid numerical methods for direct solution of initial value problems of second order ordinary differential equations are proposed in this study. Chebyshev polynomials of the first kind are used as basic function for the approximate solution. The collocation and interpolation equations are generated at both grid and off-grid points for step number $k=2$. The resulting methods are consistent and zero stable. The main predictors having the same order with the main method are developed for the implementation of the methods. The comparison shows a better performance than the existing methods.

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INTRODUCTION

In this paper, we consider numerical methods for solving second order initial value problems of ordinary differential equations of the form

$$y''(x) = (x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0, \quad (1)$$

Equation (1) arises from many physical phenomena in a wide variety of applications especially in engineering problems such as resonance vibration of machine, cooling of a body and other area of application. Many authors have worked at solving (1) numerically by reducing it to a system of first order equations [James *et.al.* (2013), Singh and Ramos (2018), Akinfenwa *et.al.* (2020), Ogunniran *et.al.* (2022)] and others. In spite of the success of this approach, it suffers some setbacks, according to [Awoyemi *et.al.* (2014), Kayode and Obarhwa (2015), Areo and Rufai (2016)], the setback are, un-economical in term of implementation cost, increased computational burden and wastage of computer time, increased dimension of the resulting systems of equations to be solved. This led many authors to attempt at solving problem (1) directly without reduction to systems of first order equation. Badmus A.M. and Yahaya Y.A (2009), Develop an implicit collocation method for direct solution of second order ordinary

differential equation directly. Awari and Abada (2014) presented a class of seven point zero stable continuous block method for solution of second order differential equation to solve (1). Kuboye *et.al.*(2018) Generalized hybrid block method for solving second order ordinary differential directly. Kayode *et.a.* (2021) develop two step continuous trigonometric ally fitted method for solving oscillatory second order (ODEs). Olanegan *et.al.* (2018) implicit hybrid points approach for solving general second order and the method is of order five. Likewise Adegboro (2022) develop a trigonometrically fitted predictor-corrector method for solving oscillatory second order ordinary differential equations without reduction to a system of first order. In this paper, efforts are made to develop a class of chebyshev polynomial to solve second order directly. The method is consistent, symmetric and of smaller error constant, and can effectively handle mildly stiff and Real life problems.

MATERIALS AND METHODS

This work considers numerical methods for solving second order initial value problems of ordinary differential equations of the form

$$y''(x) = (x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0, \quad (2)$$

using Chybyshhev polynomial of the form

$$y(x) = \sum_{i=0}^{(c+i)-1} a_i T_i(x) \quad (3)$$

as a basic function

where c and i represent the number of collocation and interpolation points respectively, (a_i) are parameters to be determined, $T_i(x)$ are continuous and differentiable terms of Chebyshev polynomial. The first and second derivatives of (2) yield

$$y'(x) = \sum_{i=0}^{(c+i)-1} a_i T'_i(x). \quad (4)$$

$$y''(x) = \sum_{i=2}^{(c+i)-1} a_i T''_i(x). \quad (5)$$

And Combining (1) and (4) gives

$$f(x, y, y') = \sum_{i=2}^{(c+i)-1} a_i T''_i(x). \quad (6)$$

Interpolation (2) at points x_{n+i} , $i = 0, r, s, 1$ and Collocating (4) at all grid points x_{n+c} , $c = 0, 1, 2$ where $r \in (0, 1)$

.The value of (a_i) were determined, using Gaussian elimination method.

. substituting the values of $a_1, a_2, a_3, \dots, a_6$ in to equation 2 to have the required continuous coefficients hybrid linear multistep method.

$$y_k(t) = \alpha_0(t)y_n + \alpha_1(t)y_{n+r} + \alpha_2(t)y_{n+s} + \alpha_3(t)y_{n+1} + h^2(\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2}) \quad (7)$$

Applying the transformation $t = \left(\frac{x-x_n+k-1}{h}\right), \frac{dt}{dx} = \frac{1}{h}$, $t \in (0, 1)$ in Obarhua (2022), to (8), its continuous coefficients α'_j 's, and β'_j 's, as functions of t obtained to be

$$\alpha_0(t) = \frac{A_0}{B_0}, \alpha_1(t) = \frac{A_1}{B_1}, \alpha_2(t) = \frac{A_2}{B_2}, \alpha_3(t) = \frac{A_3}{B_3}, \text{ while } \beta_0(t) = \frac{C_0}{B_4}, \beta_1(t) = \frac{C_1}{B_4}, \beta_2(t) = \frac{C_2}{B_4}.$$

Where

$$\begin{aligned} A_0 &= \left\{ \begin{aligned} &(-1+r-t)t(1-s+t)(11-22t-17t^2+16t^3+s^2(-47+61t+19t^2-24t^3)+2s^3(7-7t-3t^2+3t^3) \\ &+s(22-55t-t^2+16t^3)+2r^3(7+3s^3+3s^2(-3+t)-7t-3t^2+3t^3+s(-1-6t+3t^2)+r^2 \\ &(-47+6s^3(-3+t)+61t+19t^2-24t^3+s^2(49-36t+6t^2)+s(16+29t-30t^2+6t^3+r(22-55 \\ &t-t^2+16t^3+2s^3(-1-6t+3t^2)+s(-36+28t+35t^2-24t^3)+s^2(16+29t-30t^2+6t^3)) \end{aligned} \right\} \\ A_1 &= - \left\{ \begin{aligned} &t(1+t)(-1+t)^2(11-33t+16t^2)-s(1+t)^2(11-33t+16t^2)+2s^4(7-7t-3t^2+3t^3) \\ &+s^3(-61+61t+39t^2-24t^3-6t^4)+3s^2(23-23t-27t^2+7t^3+8t^4) \end{aligned} \right\} \\ A_2 &= \left\{ \begin{aligned} &-t(1+t)(-1+t)^2(11-33t+16t^2)-r(1+t)^2(11-33t+16t^2)+2r^4(7-7t-3t^2+3t^3) \\ &+r^3(-61+61t+39t^2-24t^3-6t^4)+3r^2(23-23t-27t^2+7t^3+8t^4) \end{aligned} \right\} \\ A_3 &= \left\{ \begin{aligned} &(-1+r-t)(1-s+t)(1+t)(2r^3(8+3s^3+3s^2)+2s^2(-4+t-6t^2+3t^3+s(8-9t+3t^2)) \\ &+s(1+t)(20(3-5t+2t^2)+2s^2(8-9t+3t^2)-5s(13-17t+6t^2)+r^2(-65+6s^3(-4+t) \\ &+20t+55t^2-30t^3+s^2(91-48t+6t^2)+s(-49+83t-42t^2+6t^3))+r(2s^3(8-9t+3t^2) \\ &+20(3-2t-3t^2+2t^3)+s^2(-49+83t-42t^2+6t^3)-5s(1+16t-19t^2+6t^3) \end{aligned} \right\} \\ C_0 &= \left\{ \begin{aligned} &h^2(-1+r-t)(-1+s-t)t(1+t)(66-198t+96t^2+s(-91+207t-92t^2)+s^2(25-50t+21t^2) \\ &+r^2(25-50t+21t^2+s(-34+43t-15t^2)+3s^2(3-3t+t^2))+r(-91+207t-92t^2 \\ &+s^2(-34+43t-15t^2)+s(125-193t+73t^2)/12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^2) \\ &+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3) \end{aligned} \right\} \\ C_1 &= \left\{ \begin{aligned} &h^2(-1+r-t)t(1+t)(1-s+t)(-s(1+t)(52-28t+s(-19+9t))+r^2(19+10t-9t^2+s(38)) \\ &+s^3(-61+61t+39t^2-24t^3-6t^4)+3s^2(23-23t-27t^2+7t^3+8t^4)) \end{aligned} \right\} \\ C_2 &= \left\{ \begin{aligned} &h^2(-1+r-t)(-1+s-t)t(1+t)(-s(1+t)(1-4t+s(-1+3t))+r^2(1-2t-3t^2) \\ &s(2-5t-3t^2)+3s^3(-1+t+t^2)+r(-1+3t+4t^2+s^2(2-5t-3t^2)+s(-1+5t+t^2))) \end{aligned} \right\} \\ B_0 &= \{rs(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3))\} \\ B_1 &= \left\{ \begin{aligned} &(-1+r)r(s^2(-60+65s-16s^2)+rs^2(-60+65s-16s^2)+2r^4(8+8s-12s^2) \\ &+3s^3)-r^3(65+65s-75s^2+6s^4)+3r^2(20+20s-25s^3+8s^4) \end{aligned} \right\} \\ B_2 &= \left\{ \begin{aligned} &(-1+s)s(s^2(60-65s+16s^2)+rs^2(60-65s+16s^2)-2r^4(8+8s-12s^2+3s^3)) \\ &+(r^3(65+65s-75s^2+6s^4)-3r^2(20+20s-25s^3+8s^4)) \end{aligned} \right\} \\ B_3 &= \left\{ \begin{aligned} &((-1+r)(-1+s)(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)) \\ &+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3)) \end{aligned} \right\} \\ B_4 &= \left\{ \begin{aligned} &(12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2) \\ &16s^2)-r^2(65+49s-91s^2+24s^2)) \end{aligned} \right\} \\ B_5 &= \left\{ \begin{aligned} &(12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2) \\ &16s^2)-r^2(65+49s-91s^2+24s^2)) \end{aligned} \right\} \\ B_6 &= \left\{ \begin{aligned} &(12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2) \\ &16s^2)-r^2(65+49s-91s^2+24s^2)) \end{aligned} \right\} \\ B_4 &= \{B_5 = B_6 = B_4\} \end{aligned}$$

The first derivative of the coefficients will be

$$\alpha'_0(t) = \frac{A'_0}{B_0}, \alpha'_1(t) = \frac{A'_1}{B_1}, \alpha'_2(t) = \frac{A'_2}{B_2}, \alpha'_3(t) = \frac{A'_3}{B_3}, \text{ while } \beta'_0(t) = \frac{C'_0}{B_4}, \beta'_1(t) = \frac{C'_1}{B_4}, \beta'_2(t) = \frac{C'_2}{B_4}$$

Evaluating the coefficients value and the first derivative of the coefficients at $t=1$ yield the discrete scheme.

$$y_{n+2} = \frac{1}{c_0} \alpha_0 y_n - \frac{1}{c_1} \alpha_1 y_{n+r} + \frac{1}{c_2} \alpha_2 y_{n+s} - \frac{1}{c_3} \alpha_3 y_{n+1} + \frac{h^2}{6c_4} [-\beta_0 f_n + 2\beta_1 f_{n+1} + \beta_2 f_{n+2}] \quad (8)$$

with first derivative

$$y'_{n+2} = \frac{1}{c_0} \alpha'_0 y_n - \frac{1}{c_1} \alpha'_1 y_{n+r} + \frac{1}{c_2} \alpha'_2 y_{n+s} - \frac{1}{c_3} \alpha'_3 y_{n+1} + \frac{h^2}{6c_4} [-\beta'_0 f_n + 2\beta'_1 f_{n+1} + \beta'_2 f_{n+2}] \quad (9)$$

Where

$$\alpha_0 = (-2+r)(2-s)(-12-18s+9s^2+r^2(9+21s+19s^2-12s^3)) + r(-18+3s+21s^2-8s^3) + 2r^3(-4s-6s^2+3s^3))$$

$$\alpha_1 = 2(24+24s-36s^2+9s^3)$$

$$\alpha_2 = 2(24+24r-36r^2+9r^3)$$

$$\alpha_3 = 2(-2+r)(-2+s)(2s(-10s+4s^2)+r^2(-20-2s+49s^2-18s^3)+2r^3(4+2s-9s^2+3s^3)+r(-20s-2s^2+4s^3))$$

$$\beta_0 = h^2(-2+r)(-2+s)(-36+24s-4s^2+r(24+5s-6s^2)+r^2(-4-6s+3s^2))$$

$$\beta_1 = h^2(-2+r)(2-s)(-2(24-10s)s+r^2(20+30s-15s^2)+r(-48-52s+30s^2)) \quad \beta_2 = h^2(-2+r)(-2+s)(-2s(-3+2s)+r(6+5s-6s^2)+r^2(-4-6s+3s^2))$$

$$C_0 = rs(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3))$$

$$C_1 = (-1+r)r(s^2(-60+65s-16s^2)+rs^2(-60+65s-16s^2)+2r^4(8+8s-12s^2+3s^3)-r^3(65+65s-75s^2+6s^4)+3r^2(20+20s-25s^3+8s^4))$$

$$C_2 = (-1+s)s(s^2(60-65s+16s^2)+rs^2(60-65s+16s^2)-2r^4(8+8s-12s^2+3s^3)+r^3(65+65s-75s^2+6s^4)-3r^2(20+20s-25s^3+8s^4))$$

$$C_3 = (-1+r)(-1+s)(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3))$$

$$C_4 = 6(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3))$$

and

$$\alpha'_0 = 2r^3(3s^2-12s+8)+r^2(6s^3-48s^2+107s-49)+r(-24s^3+107s^2-114s-5)+16s^3-49s^2-5s+60$$

$$2r^3(3s^3-12s^2+8s+8)+r^2(-24s^3+91s^2-49s-65)+r(16s^3-49s^2-5s+60)+16s^3-65s^2+60s$$

$$\alpha'_1 = rs(2r^3(3s^3-15s^2+20s)+r^2(-30s^3+145s^2-180s)+r(40s^3-180s^2+200s\alpha'_2)=16s^4-65s^3+60s^2$$

$$\alpha'_3$$

$$\beta'_0$$

$$\beta'_1 \beta'_2$$

$$= 16r^4 - 65r^3 + 60r^2$$

$$= h^2 r s f_n (r^2 (21s^2 - 92s + 96) + r (-92s^2 + 391s - 390) + 96s^2 - 390s + 360)$$

$$= h^2 r s f_{n+1} (r^2 (28s - 9s^2) + r (28s^2 - 80s))$$

$$= h^2 r s (r^2 (4s - 3s^2) + r (4s^2 - 5s))$$

$$C'_0 = s(2r^3(3s^3-12s^2+8s+8)-r^2(24s^3-91s^2+49s+65)+r(16s^3-49s^2-5s+60)+s(16s^2-65s+60))$$

$$C'_1 = (r-1)(s-1)(2r^3(3s^3-12s^2+8s+8)-r^2(24s^3-91s^2+49s+65)+r(16s^3-49s^2-5s+60)+s(16s^2-65s+60))$$

$$C'2 = (r-1)r(2r^4(3s^3 - 12s^2 + 8s + 8) - r^3(6s^4 - 75s^2 + 65s + 65) + 3r^2(8s^4 - 25s^3 + 20s + 20) + rs^2(-16s^2 + 65s - 60) + s^2(-16s^2 + 65s - 60))$$

$$C'3 = (s-1)s(-2r^4(3s^3 - 12s^2 + 8s + 8) + r^3(6s^4 - 75s^2 + 65s + 65) - 3r^2(8s^4 - 25s^3 + 20s + 20) + rs^2(16s^2 - 65s + 60) + s^2(16s^2 - 65s + 60))$$

$$C'4 = (r-1)r(2r^4(3s^3 - 12s^2 + 8s + 8) - r^3(6s^4 - 75s^2 + 65s + 65) + 3r^2(8s^4 - 25s^3 + 20s + 20) + rs^2(-16s^2 + 65s - 60) + s^2(-16s^2 + 65s - 60))$$

in order to test the properties of the continuous methods (8), we choose the values of r and s at various points in the interval of $(0,1)$ to obtain our derived discrete hybrid methods. Therefore, for the purpose of testing, the values of r and s are taken to $\frac{2}{5}, \frac{3}{5}$

Respectively to have

$$y_{n+2} = \left[\frac{2927204}{369671} y_n - \frac{17828125}{369671} y_{n+\frac{2}{5}} + \frac{1850000}{369671} y_{n+\frac{3}{5}} - \frac{3229408}{369671} y_{n+1} \right] - \frac{h^2}{1109013} [247576 f_n - 1531040 f_{n+1} - 674 f_{n+2}] \quad (10)$$

The order p and the principal error constant c_{p+2} of (11) are $p = 5$ and $c_{p+2} = 8.4904 \times 10^{-4}$ respectively and its first derivative is.

$$y'_{n+2} = \left[\frac{43812599}{2218026} y_n - \frac{266434375}{2218026} y_{n+\frac{2}{5}} + \frac{142287500}{1109013} y_{n+\frac{3}{5}} - \frac{30976612}{1109013} y_{n+1} \right] - \frac{h^2}{369671} [208021 f_n - 1024204 f_{n+1} - 101136 f_{n+2}] \quad (11)$$

The order p and the principal error constant c_{p+2} of (12) are $p = 5$ and $c_{p+2} = 2.2865 \times 10^{-3}$ respectively.

Implementation of the Method

Implementation of the discrete method (10) obtained from (8) for problem (1) requires the generation of some starting values. This is obtained in predictor-corrector mode of the same order of accuracy. The following

explicit Predictor scheme and its derivative, of the same order with the corrector scheme, are obtained using the same procedure in section 2 for y_{n+2} and y'_{n+2} .

$$y_{n+2} = \frac{1}{10707} [4336332 y_n - 10609375 y_{n+\frac{2}{5}} + 510000 y_{n+\frac{3}{5}} - 11623361 y_{n+1}] - \frac{h^2}{32121} [168840 f_n - 1685600 f_{n+\frac{2}{5}} - 161056 f_{n+1}] \quad (12)$$

$$y'_{n+2} = \frac{1}{21414} [39367039 y_n - 97559375 y_{n+\frac{2}{5}} + 47975000 y_{n+\frac{3}{5}} - 10217336 y_{n+1}] - \frac{h^2}{3569} [85997 f_n - 428008 f_{n+\frac{2}{5}} - 68244 f_{n+1}] \quad (13)$$

The principal error constant of (13) and (14) are $c_{p+2} = 2.998 \times 10^{-3}$ and $c_{p+2} = 1.5025 \times 10^{-2}$ respectively. The schemes (12) and (13) above has the same order $p = 5$.

Other explicit schemes were also generated to evaluate other starting values and Taylor's series expansion was used to evaluate the values for y_{n+r} , that is

$$y_{n+r} = y_n + (rh)y'_n + \frac{(rh)^2}{2!} f_n + \frac{(rh)^3}{3!} \frac{\partial f_n}{\partial x_n} + y'_n \frac{\partial f_n}{\partial y_n} + f_n \frac{\partial f_n}{\partial y'_n} \} + 0(h^4) \quad (14)$$

$$y'_{n+r} = y'_n + (rh)f_n + \frac{(rh)^2}{2!} \frac{\partial f_n}{\partial x_n} + y'_n \frac{\partial f_n}{\partial y_n} + f_n \frac{\partial f_n}{\partial y'_n} \} + 0(h^4) \quad (15)$$

Analysis of the Method

Order and Error Constant of the Method

let the linear difference operator L associated with the continuous method be defined as

$$L[y(x); h] = \sum_{j=0}^k [\alpha_j y(x + rh) - h^n \beta_j f(x + rh), \dots, yv(x + rh)]; r = 1, 2, \dots, m \quad (16)$$

$y(x)$ represent an arbitrary test function that is continuously differentiable in the interval $[a, b]$. Taylor series expansion of $y(x + rh)$ and $y''(x + rh)$, $r = 1, 2, \dots, m$ about x_n and collecting the like terms in h and y yields

$$L[y(x); h] = c_{-0} y(x) + c_{-1} h y'(x) + c_{-2} h^2 y''(x) + \dots + c_{-p} h^p y^{(p)}(x) \quad (17)$$

The difference operator L and the continuous multi step method are said to be of order

$$c_0 = c_1 = c_2 = \dots = c_p = 0, c_{p+2} \neq 0$$

The term C_{p+2} is called the error constant and it implies that the local truncated error (LTE) is given by $(LTE) = c_{p+1} h^{p+1} y^{(p+1)}(x_n) + 0(h^{p+3})$. (Lambert 1973)

$$y_{n+2} = \left[\frac{2927204}{369671} y_n - \frac{17828125}{369671} y_{n+\frac{2}{5}} + \frac{1850000}{369671} y_{n+\frac{3}{5}} - \frac{3229408}{369671} y_{n+1} \right] - \frac{h^2}{1109013} [247576f_n - 1531040f_{n+1} - 67424f_{n+2}]$$

$$y'_{n+2} = \left[\frac{43812599}{2218026} y_n - \frac{266434375}{2218026} y_{n+\frac{2}{5}} + \frac{142287500}{1109013} y_{n+\frac{3}{5}} - \frac{30976612}{1109013} y_{n+1} \right] - \frac{h^2}{369671} [208021f_n - 1024204f_{n+1} - 101136f_{n+2}]$$

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0, C_7 = \frac{4708}{5545065} \approx 8.4904 \times 10^{-4}$$

Hence the method is of order 5 with error constant $C_7 = 8.4904 \times 10^{-4}$ and for the first derivative

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0, C_7 = \frac{13313017}{5822318250} \approx 2.2865 \times 10^{-3}$$

Hence the first derivative of the method is of order 5 with error constant $C_7 = 2.2865 \times 10^{-3}$

The main predictor

$$y_{n+2} = \frac{1}{10707} \left[4336332y_n - 10609375 y_{n+\frac{2}{5}} - 510000 y_{n+\frac{3}{5}} + 1162336y_{n+1} \right] - \frac{h^2}{32121} \left[168840f_n - 1685600f_{n+\frac{2}{5}} - 161056f_{n+1} \right]$$

with its first derivative

$$y'_{n+2} = \frac{1}{21414} \left[39367039y_n - 97559375 y_{n+\frac{2}{5}} - 47975000 y_{n+\frac{3}{5}} + 10217336y_{n+1} \right] - \frac{h^2}{3569} \left[85997f_n - 842800f_{n+\frac{2}{5}} - 68244f_{n+1} \right]$$

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0, C_7 = \frac{60188}{20075625} \approx 2.9981 \times 10^{-3}$$

Hence the main predictor of the method is of order 5 with error constant $C_7 = 2.9981 \times 10^{-3}$

and

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0, C_7 = \frac{4223029}{281058750} \approx 1.5025 \times 10^{-2}$$

Hence the first derivative of the main predictor of the method is of order 5 with error constant

$$C_7 = 1.5025 \times 10^{-2}$$

Consistency

The linear multistep method for [ivp] is said to be consistent if the following conditions are satisfied:

i the order of the scheme must be greater than or equal to 1 ($p \geq 1$)

ii $\sum_{j=0}^k \alpha_j = 0$, α_j 's are the coefficients of the first characteristics polynomial

iii $\rho(r) = \rho'(r) = 0$ for $r=1$

iv $\rho''(r) = 2!\sigma(r)$ for $r=1$. $\rho(r)$ and $\sigma(r)$ are first and second characteristic polynomial respectively.

consistency of the methods

$$y_{n+2} = \left[\frac{2927204}{369671} y_n - \frac{17828125}{369671} y_{n+\frac{2}{5}} + \frac{1850000}{369671} y_{n+\frac{3}{5}} - \frac{3229408}{369671} y_{n+1} \right] - \frac{h^2}{1109013} [247576f_n - 1531040f_{n+1} - 67424f_{n+2}]$$

condition (i) is satisfied since the scheme is of order 5

condition (ii) is satisfied since $\sum_{j=0}^k \alpha_j = 0$

$$\rho(r) = \left[r^2 - \frac{17828125r^{2/5}}{369671} + \frac{185000004r^{3/5}}{369671} - \frac{3229408r^1}{369671} + \frac{2927204}{369671} \right] = 0$$

$$\rho(r) = \left[r^2 - \left(\frac{17828125r^{2/5}}{369671} + \frac{185000004r^{3/5}}{369671} - \frac{3229408r^1}{369671} + \frac{2927204}{369671} \right), r \rightarrow 1 \right] = 0$$

$$\rho'(r) = \left[- \left(\frac{11100000}{369671r^{2/5}} + \frac{7131250}{369671r^{3/5}} + 2r + \frac{3229408}{369671} \right), r \rightarrow 1 \right] = 0$$

condition (iv) is satisfied since $\rho''(r) = 2!\sigma(r)$ for $r=1$. $\rho(r)$ and $\sigma(r)$ are first and second characteristic polynomial respectively.

$$\rho''(r) = \frac{4278750}{369671r^{8/5}} + \frac{4440000}{369671} + 2 \text{ and } \sigma(r) = \frac{1568r^2}{25791} + \frac{1531040r^1}{1109013} - \frac{247576}{1109013}$$

therefore, if $r=1$, then $\rho''(r) = 2! \sigma(r) = \frac{20944}{8579}$

hence, the four conditions of consistency are satisfied, the 2-step method is consistent.

Zero Stability

The linear multi step Method is Said to be Zero Stable if no Root of the First Characteristic Polynomial $p(r)$ has a Modulus Greater than one and if Every Root of Modulus one has Multiplicity not more than one.

The first characteristic polynomial

$$\rho(r) = \left[r^2 - \left(\frac{17828125r^{2/5}}{369671} + \frac{18500000r^{3/5}}{369671} - \frac{3229408r^1}{369671} + \frac{2927204}{369671} \right), r \rightarrow 0 \right] = 1$$

Equating to zero and solving for r gives

$$r = 1$$

The root r of 2-step method for which $|r| = 1$ is simple, hence the method is zero stable

Convergence

Theorem 1: convergence - The necessary and sufficient condition for a linear multi step method to be convergent is for it to be consistent and zero stable. From the theorem above, the new hybrid predictor - corrector method is convergent.

Region of Absolute Stability of the Methods

considering the stability polynomial of the linear multistep method defined by ρ and σ , $\pi(r, h^-) = \rho(r) - h^- \sigma(r) = 0$, where $h^- = h\lambda$.

Applying equation (15) and (16) to our method we have

$$\bar{h}(r) = \frac{r^2 - \left(\frac{17828125r^{2/5}}{369671} + \frac{18500000r^{3/5}}{369671} - \frac{3229408r^1}{369671} + \frac{2927204}{369671} \right)}{\frac{1568r^2}{25791} + \frac{1531040r^1}{1109013} - \frac{247576}{1109013}} \quad (20)$$

By setting

$r = \exp^{i\theta}$, $0 \leq \theta \leq \pi$, where $e^{i\theta} = \cos\theta + i\sin\theta$, then in equation (3.112), we have

$$\bar{h}(r) = \frac{e^{2i\theta} - \left(-\frac{17828125e^{2/5}}{369671} + \frac{18500000e^{3/5}}{369671} - \frac{3229408e^{i\theta}}{369671} + \frac{2927204}{197506} \right)}{\frac{1568e^{2i\theta}}{25791} + \frac{1531040e^{i\theta}}{1109013} - \frac{27756}{1109013}} \quad (21)$$

$$\bar{h}(\theta) = \frac{\cos(2\theta) + i\sin(2\theta) - \left(-\frac{3229408(\cos(\theta) + i\sin(\theta))}{369671} - \frac{17828125\cos(\frac{2\theta}{5}) + i\sin(\frac{2\theta}{5})}{369671} + \frac{18500000\left(\cos(\frac{3\theta}{5}) + i\sin(\frac{3\theta}{5})\right)}{369671} + \frac{2927204}{369671} \right)}{\frac{1531040(\cos\theta + i\sin\theta)}{1109013} + \frac{1568(\cos\theta + i\sin\theta)}{25791} - \frac{247576}{1109013}} \quad (22)$$

$$\bar{h}(\theta) = \frac{\frac{17828125(\cos\theta + i\sin\theta)}{369671} + \frac{3229408(\cos(\theta) + i\sin(\theta))}{369671} + \cos(2\theta) + i\sin(2\theta) + \frac{18500000\left(\cos(\frac{3\theta}{5}) + i\sin(\frac{3\theta}{5})\right)}{369671} + \frac{2927204}{369671}}{\frac{1531040(\cos\theta + i\sin\theta)}{1109013} + \frac{1568(\cos\theta + i\sin\theta)}{25791} - \frac{247576}{1109013}} \quad (23)$$

$$\bar{h}(\theta) = \frac{3(17828125 \cos(\frac{2\theta}{5}) + i\sin(\frac{2\theta}{5}) - 18500000 \cos(\frac{3\theta}{5}) + i\sin(\frac{3\theta}{5}))}{56(27340 \cos(\theta) + i\sin(\theta) + 1204 \cos(2\theta) + i\sin(2\theta) - 4421)} + \frac{3229408 \cos(\theta) + i\sin(\theta) + 369671 \cos(2\theta) + i\sin(2\theta) - 2927204}{56(27340 \cos(\theta) + i\sin(\theta) + 1204 \cos(2\theta) + i\sin(2\theta) - 4421)} \quad (24)$$

multiplying by conjugate gives $\bar{h}(\theta) = x(\theta) + iy(\theta)$ and considering the real part gives

$$\bar{h}(\theta) = \frac{3(17828125 \cos(\frac{2\theta}{5}) - 18500000 \cos(\frac{3\theta}{5}) - 3229408(\theta) + 369671 \cos(2\theta) - 2927204)}{56(27340 \cos(\theta) + 1204 \cos(2\theta) - 4421 + i56(27340 \sin(\theta) + 1204 \sin(2\theta) - 4421))} \quad (25)$$

Definition

A linear multistep method is said to be absolutely stable for a given value h^- , if for all roots of $|r_s|$ of $\pi(r, h) = \rho(r, h) - h^- \sigma(r) = 0$ satisfies $|r_s| = 1$, $s = 1, 2, \dots, k$ and absolutely unstable for that h^-

otherwise. The region of absolute stability is the region where the stability of the method is guaranteed. We shall adopt the boundary locus method proposed by Lambert (1973). The boundary locus method is given by

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \quad (18)$$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomial respectively, taking

$$r = e^{i\theta} = \cos\theta + i\sin\theta \quad (19)$$

$$\bar{h}(\theta) = + \frac{3(17828125 \cos(\frac{2\theta}{5}) - 18500000 \sin(\frac{3\theta}{5}) + 3229408 \cos(\theta) + 369671 \sin(2\theta) - 2927204)}{56(27340 \cos(\theta) + 1204 \cos(2\theta) - 4421)} \quad (26)$$

Evaluating at intervals of $(0^0, 180^0)$ gives $(-9.647, 0, 0)$

Numerical Experiments

The method is applied to solve the following linear, nonlinear and real life second order initial value problems of ordinary differential equations directly without reduction to system of first order equations.

Problem 1: Consider a Linear Second Order Problem

$$y'' = y', y(0) = 0, y'(0) = -1h = 0.01$$

$$\text{Exact solution: } y(x) = 1 - e^x$$

Source: Adeyefa & Olagunju (2021)

Table 1: Comparison of Absolute Errors of Problem 1 with Existing Methods

X-value	Exact-solution	Computed-solution	Error in New Method	Error in Adeyefa and Olagunju (2021)
0.100	-0.105170918075647710	-0.105170918073591250	2.056466e-012	1x10e-10
0.200	-0.221402758160169850	-0.221402758151950210	8.219647e-012	1x10e-10
0.300	-0.349858807576003180	-0.349858807556600930	1.940226e-011	0
0.400	-0.491824697641270570	-0.491824697604673840	3.659673e-011	2x10e-10
0.500	-0.648721270700128640	-0.648721270639189380	6.093925e-011	4x10e-10
0.600	-0.822118800390509550	-0.822118800296757430	9.375212e-011	4x10e-10
0.700	-1.013752707470477500	-1.013752707333933900	1.365437e-010	3x10e-10
0.800	-1.225540928492468800	-1.225540928301450400	1.910183e-010	3x10e-10
0.900	-1.459603111156951200	-1.459603110897772300	2.591789e-010	3x10e-10
1.000	-1.718281828459047300	-1.718281828115713900	3.433334e-010	1x10e-10

Problem 2: Consider a Nonlinear Second Order Problem

$$y'' = x(y')^2, y(0) = \frac{1}{2}, h = 0.03125$$

$$\text{Exact solution: } y(x) = 1 + \frac{1}{2} \log\left(\frac{2+x}{2-x}\right)$$

Source: Olanegan et al. (2018)

TABLE 2: Comparison of absolute errors of Problem 2 with exiting methods

X-value	Exact-solution	Computed-solution	Error in new Method	Error in Olanegan et al. (2018)
0.1	1.050041729278491400	1.050041821975586500	9.269710e-008	7.476427e-06
0.2	1.100335347731075300	1.100335713636875200	3.659058e-007	2.939419e-05
0.3	1.151140435936466500	1.151141253691998000	8.177555e-007	6.480165e-05
0.4	1.202732554054081600	1.202733999164715700	1.445111e-006	5.127120e-04
0.5	1.255412811882994600	1.255415053925528200	2.242043e-006	1.127905e-05
0.6	1.309519604203111900	1.309522802065695000	3.197863e-006	1.724976e-04
0.7	1.365443754271397100	1.365448048062983600	4.293792e-006	2.431027e-04
0.8	1.423648930193603500	1.423654427484050800	5.497290e-006	3.238270e-04
1.0	1.549306144334058600	1.549314105526519800	7.961192e-006	4.139307e-04

Problem 3: Resonance Vibration of a Machine

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger moves vertically up and down by a fly wheel spinning at constant set speed. The constant rotational speed of the fly wheel makes the impact force on the sheet metal, and therefore the supporting base, intermittent and cyclic. The bearing base on which the metal sheet is situated has a mass, $M = 2000 \text{ kg}$. The force acting on the base follows a function: $F(t) = 2000 \sin(10t)$, in which $t = \text{time}$ in seconds. The base is supported by an elastic pad with an equivalent

spring constant $k = 2 \times 10^5 \text{ N/m}$. Determine the differential equation for the instantaneous position of the base $y(t)$ if the base is initially depressed down by an amount 0.1 m . [Kayode and Adegboro (2018)].

Solution

The mass spring system above is modeled as differential equation: The bearing base mass = 2000 kg

Spring constant $k = 2 \times 10^5 \text{ N/M}$

Force (ma) on the metal sheet = $m^d 2y = my''$

i.e $ma = my'' = 2000 \sin(10t)$: where $a = y''$

Initial conditions on the system are

$$y(t_0) = y_0; y'(t_0) = y'(0) = 0.1, y(0) = 0$$

Therefore, the governing equation for the instantaneous position of the base $y(t)$ is given by

$$My'' + ky = F(t); y(t_0) = y(0) = 0.1, y(0) = 0$$

Theoretical Solution

$$y(t) = \frac{1}{10} \cos(10t) + \sin(10t) - \frac{t}{20} \cos(10t) \text{ Source: Kayode et al. (2021)}$$

Table 3: Comparison of Absolute Errors of Problem 3 with Existing Methods

X-value	Exact-solution	Computed-solution	Error in New method	Error in Kayode et al., (2021)
0.01	0.099805157812128037	0.099805130172962331	2.763917e-008	4.98251761e-07
0.02	0.098788120701908358	0.098787939006475867	1.816954e-007	1.99750245e-06
0.03	0.096917121440198789	0.096916589135351949	5.323048e-007	4.49460223e-06
0.04	0.093137423045576082	0.093136011635916560	1.411410e-006	7.99469593e-06
0.05	0.089381288936266456	0.089378837567449465	2.451369e-006	1.24926327e-05
0.06	0.084879800636699518	0.084875929133199080	3.871504e-006	1.79955544e-05
0.07	0.079679648376031073	0.079673936029610593	5.712346e-006	2.44963093e-05
0.08	0.073833247429221927	0.073825241230734195	8.006198e-006	3.20040361e-05
0.09	0.065133042073048070	0.065121233642837062	1.180843e-005	4.05095821e-05
0.10	0.058010749973883932	0.057995516097657018	1.523388e-005	5.00240831e-05

Problem 4: Mass Spring Motion

A 128lb weight is attached to a spring having a spring constant of 64lb/ft. The weight is started in motion with no initial velocity by displacing it 6inches above the equilibrium position and by simultaneously applying to the weight an external force $F(t) = 8\sin(4t)$. Assuming no air resistance, compute the subsequent motion of the weight at $t : 0.01$ to 0.10 . we model the problem into a

mathematical equation of the motion on the weight attached to the spring.

$m = 128, k = 64, b = 0$ and $F(t) = 8\sin(4t)$. Upon simplification, the problem boils down

$$\text{to } x'' + 16x = 2\sin(4t), x(0) = -\frac{1}{2}, x'(0) = 0$$

with the exact solution $x(t) = \frac{1}{2} \cos 4t + \frac{1}{16} \sin 4t + \frac{1}{4} t \cos 4t$
Source: Skwame et al. (2017)

Table 4: Comparison of Absolute Errors of Problem 4 with Existing Methods

X-value	Exact-solution	Computed-solution	Error in New method	Error in Kayode and Adeyeye (2013)
0.02	-0.498390193309749540	-0.498390897758998810	7.044492e-007	0.82e-06
0.03	-0.496368369740279650	-0.496369713992800050	1.344253e-006	0.31e-05
0.04	-0.493528526608179360	-0.493530635122523260	2.108514e-006	0.65e-05
0.05	-0.489867287968945020	-0.489870245387725500	2.957419e-006	0.66e-05
0.06	-0.485382642897099370	-0.485386493411369630	3.850514e-006	0.11e-04
0.07	-0.480073961290566790	-0.480078708134076940	4.746844e-006	1.80e-04
0.08	-0.473942007364361870	-0.473947612439779230	5.605075e-006	0.26e-04
0.09	-0.466988950792027790	-0.466995334430852170	6.383639e-006	0.37e-04
0.1	-0.459218375457223990	-0.459225416314890560	7.040858e-006	0.51e-04

CONCLUSION

The chebyshev polynomial of the first kind collocation method was used to develop a two-step continuous hybrid method. The method obtained was used to solve second order ordinary differential equations. The new method compared favorably in terms of accuracy with the existing methods of higher order and step number. Our future research will be focused on more introductions of grid and off-grid points to enhance global error estimations.

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