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Numerical Solution of Second-Order Differential Equations Using the Haar Wavelet Technique

***Hammed Ayobami Haruna, Olutunde Samuel Odetunde and Sefiu Adekunle Onitilo**

Department of Mathematical Sciences, Olabisi Onabanjo University, Ago-Iwoye, Ogun State, Nigeria

*Corresponding Author's Email: haruna.hammed@oouagoiwoye.edu.ng

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ABSTRACT

This paper studied the solution of second order differential equations numerically by using the Haar wavelet operational matrix approach. Haar wavelet characterized by their piecewise constant properties and orthogonality are used for this work due to their computational efficiency, minimal memory requirements and ease of evaluation. Unlike higher order wavelets such as Daubechies or Legendre wavelets, Haar wavelets offer a simple but yet powerful approach particularly well-suited for problems with discontinuities or sharp gradients. The methodology involves transforming the given differential equation into a system of algebraic equations through the use of Haar wavelet operational matrices. This transformation significantly simplifies the numerical computation by reducing the problem to the determination of a finite set of wavelet coefficients. These coefficients are then solved manually. The approach is not only efficient, but also extremely accurate. The numerical experiments carried out in this study reveal that the Haar wavelet-based solutions strongly agree with the corresponding exact solutions, thereby confirming the validity and robustness of the methods.

INTRODUCTION

Wavelet analysis is a novel numerical idea that allows one to represent a function in terms of space localized basic functions known as wavelets. Wavelets became known when they were first introduced in the early 1980s. It has piqued the curiosity mathematical community as well as members of numerous disciplines where wavelets have shown great promise. Among this interest are the emergence of many books on the topic and numerous numbers of research articles (Hus09). The main idea behind the discovery of wavelet and wavelet transform techniques is that the Fourier transform is useful for frequency analysis but classical Fourier basis functions are global and therefore have limited localization in time and space. Wavelets provide simultaneous localization in position, time and frequency.

Alfred (1910) introduced a series of square waves with magnitude -1 and 1, with zeros at other points, which we referred to as Haar wavelet (Walnut, 2004). Among all these wavelet types Haar wavelet is the simplest orthonormal wavelet. (Chen and Hsiao, 1997) have gained recognition, due to their useful contributions in wavelet. They generated a novel Haar function and used it to solve a lumped and distributed parameter system.

Lepik (2007) and Lepik (2008) introduced Haar wavelet methods differential equations solution techniques. (Hariharan and Kannan, 2010). (Berwal *et al*. 2013) solving the telegraph equation with the Haar wavelet.

Although, Haar wavelet operational matrix methods have been successfully applied to various differential equations, there is need for a clearly formulated and quantitatively validated framework for solving second order differential equations using the highest derivative approximation and successive integration of the Haar expansion. In particular, existing approach do not sufficiently emphasize the verification of the operational matrix, error analysis and convergence behavior as the Haar resolution increases.

This work presents a computation approach for solving the differential equation of second order. This method reduced the problem using the Haar wavelet operational matrix to system of algebraic equations by expanding the terms which have maximum derivatives which gives the equation Haar functions and unknown coefficients. The integration of Haar wavelet is used, because the integration of the Haar functions vector is continuous function, the solution derived is also continuous. This method has less computational time and it is simple.

MATERIALS AND METHODS

The Haar Wavelet Operational Matrix for Ordinary Differential Equation

The Haar wavelet is defined as $t \in [0,1]$. The orthogonal set of Haar function are defined in the interval [0,1) by $h_0 = 1$

$$h_i = \begin{cases} 1, \frac{k-1}{2^j} \le t < \frac{k-0.5}{2^j} \\ -1 \ \frac{k-0.5}{2^j} \le t < \frac{k}{2^j} \\ 0 \ elsewhere \end{cases} \quad (1)$$

where $i = 1,2,3,\dots m-1, m = 2^M$ and M is a positive integer j and k represent the integer decomposition of the index i

Any function $f(t) \in L^2([0,1])$ can be expanded in Haar series

$$f(t) = \sum_{i=0}^{\infty} C_i h_i(t) \quad (2)$$

where $C_i, i = 0,1,2,\dots$ is the Haar coefficient, which is given by

$$C_i = 2^j \int_0^1 f(t)\, h_i(t) dt \quad (3)$$

$$f(t) \simeq \sum_{i=0}^{m-1} C_i h_i(t) = C_m H_m(t) = f(t) \quad (4)$$

where $m = 2^j$. The Haar coefficient vector C_m and function vector $H_m(t)$ is defined as

$$C_m \triangleq [C_0 C_1 C_2 \dots C_m]^T. \quad (5)$$

The collocation points is given as

$$t_k = \frac{(2k-1)}{2m}, K = 1,2,\dots m \quad (6)$$

The m-square matrix $\Phi_{m\times m}$ is defined as

$$\Phi_{m\times m} \triangleq [H_m(\tfrac{1}{2m}), H_m(\tfrac{3}{2m}), H_m(\tfrac{5}{2m}), \dots, H_m(\tfrac{2m-1}{2m})] \quad (7)$$

The Wavelet Operational Matrix

The integration of the $H_m(t)$, were approximated by (Chen and Hsiao, 1997) as,

$$\int_0^t H_m(\tau)d\tau \cong P^1_{m\times m} H_m(t) \quad (8)$$

where $P^1_{m\times m}$ is the Haar wavelet operational matrix of integration.

$$\int_0^t P^{m-1}_{m\times m} H_m(\tau)d\tau \cong P^n_{m\times m} H_m(t) \ n = 2,3,\dots \quad (9)$$

We then define an m-set of block function (Lepik, 2006) as;

$$\lambda_i(t) = \begin{cases} 1, \frac{i}{m} \le t < \frac{i+1}{m} \\ 0, elsewhere \end{cases} \quad (10)$$

where, $i = 1,2,3,\dots(m-1)$

The function $\lambda_i(t)$ is disjointed and orthogonal.i.e ,

$$\lambda_i(t)\lambda_l(t) = \begin{cases} 0, i \ne l \\ \lambda_i(t), i = l \end{cases} \quad (11)$$

$$\int_0^1 \lambda_i(\tau)\lambda_l(\tau)\, d\tau = \begin{cases} 0, i \ne l \\ \frac{1}{m}, i = l \end{cases} \quad (12)$$

Haar wavelet functions are piecewice constant and may be expanded into an m-term block pulse functions.

$$H_m(t) = \Phi_{m\times m}\lambda_m(t) \quad (13)$$

where, $\lambda_m(t) \triangleq [\lambda_0(t), \lambda_1(t) \dots \lambda_{m-1}(t)]$ (14)

(Kilicman and Al Zhour, 2007), have given the block pulse operational matrix B^n as follows

$$(I^n\lambda_m)(t) \simeq B^n\lambda_m \quad (15)$$

where I^n shows the nth integration of function

$$B^n = \frac{1}{m^n}\frac{1}{(n+1)!}\begin{pmatrix} 1 & \theta_1 & \theta_2 & \dots & \theta_{m-1} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-2} \\ 0 & 0 & 1 & \dots & . \\ 0 & 0 & 0 & \dots & . \\ 0 & 0 & 0 & \ddots & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} n \in I \quad (16)$$

where $\theta_k = (k+1)^{n+1} - 2k^{n+1} + (k-1)^{n+1}$ (17)

we can now proceed to derive the Haar wavelet operational matrix for general integration.

$$(B^n H_m)(t) \simeq P^n_{m\times m} H_m(t) \quad (18)$$

From equation (15) and (16) we get

$$(B^n H_m)(t) \simeq \mathrm{H}_{m\times m} B^n \mathrm{H}^{-1}_{m\times m} \quad (19)$$

RESULTS AND DISCUSSION

The Application of Haar Wavelet Operational Matrix

In this section, we will use the operational matrix of Haar wavelet to find the numerical solution of second order ordinary differential equation.

Example 1

Consider an ordinary differential equation with constant coefficient.

$$\sigma'' + \sigma' + 5\sigma = f(t) \quad (20)$$

$f(t) = 3e^{-t}\sin t$ subject to $\sigma(0) = 0, \sigma'(0) = 1$

The exact solution of equation (20) is

$$\sigma = e^{-\frac{t}{2}}\left[\frac{-3}{17}cos\left(\frac{\sqrt{17}}{2}t\right) + \frac{3}{17\sqrt{19}}sin\left(\frac{\sqrt{19}}{2}t\right)\right] + e^{-t}\left[\frac{12}{17}sint + \frac{3}{17}cost\right]$$

Assume that,

$$\sigma''(t) = C_m^T H_m(t) \quad (21)$$

Integrate equation (21) with respect to t from 0 to t using the initial conditions we have,

$$\sigma'(t) - \sigma'(0) = C_m^T P^1_{m\times m} H_m(t) \quad (22)$$

$$\sigma'(t) - 1 = C_m^T P^1_{m\times m} H_m(t) \quad (23)$$

$$\sigma'(t) = C_m^T P^1_{m\times m} H_m(t) + 1 \quad (24)$$

Integrate equation (24) with respect to t from 0 to t gives

$$\sigma(t) - \sigma(0) = C_m^T P^2_{m\times m} H_m(t) + 1 \quad (25)$$

$$\sigma(t) = C_m^T P^2_{m\times m} H_m(t) + t \quad (26)$$

$f(t)$ will be expanded by Haar function as

$$f(t) = \rho_m^T H_m(t) \quad (27)$$

$f_m^T(t)$ is a known constant vector

insert equation (21), (24), (26) and (27) into (20), gives

$$C_m^T H_m(t) + [C_m^T P^1_{m\times m} H_m(t) + 1] + 5[C_m^T P^2_{m\times m} H_m(t) + t] = 3\rho_m^T H_m(t) \quad (28)$$

$$C_m^T H_m(t) + C_m^T P^1_{m\times m} 5[C_m^T P^2_{m\times m} H_m(t)] = 3\rho_m^T H_m(t) - 1 - 5t \quad (29)$$

Equation (29) is the algebraic form of equation (20). therefore, according to (Naresh *et al.* 2014) solve equation (28) manually using $m = 4$ by applying the following steps.

Construct the Haar matrix $\mathrm{H}_{4\times 4}$

Define the function and compute the integrals

Build operational matrices P

Compute ρ_m vector for $f(t) = 3e^{-t}\sin t - 1 - 5t$

Substitute the above stated result into equation (29)

Solve for C_m

Step 1

Let's define the sampling points from equation (6)

$$t_k = \frac{(2k-1)}{8}, \quad (30)$$

$$t = \left[\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right] = [0.125, 0.375, 0.625, 0.875] \quad (31)$$

Now evaluate each Haar function at these points

$h_0(t)$ on [0,1]

$$h_1(t) = \begin{cases} 1, t \in [0, 0.5) \\ -1, \ t \in [0.5, 1) \end{cases}$$

$$h_2(t) = \begin{cases} 1, t \in [0, 0.25) \\ -1, \ t \in [0.25, 0.5) \end{cases}$$

$$h_3(t) = \begin{cases} 1, t \in [0.5, 0.75) \\ -1, \ t \in [0.75, 1) \end{cases} \quad (32)$$

$$\mathrm{H}_{4\times 4} = \begin{bmatrix} h_0(0.125) & h_0(0.375) & h_0(0.625) & h_0(0.875) \\ h_1(0.125) & h_1(0.375) & h_1(0.625) & h_1(0.875) \\ h_2(0.125) & h_2(0.375) & h_2(0.625) & h_2(0.875) \\ h_3(0.125) & h_3(0.375) & h_3(0.625) & h_3(0.875) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (33)$$

Step 2

Approximate ρ_m coefficients

T	$f(t) = 3e^{-t}\sin t - 1 - 5t$ $f(t) \simeq 3e^{-t}\sin t - 1 - 5t$
0.125	−1.294925
0.375	−2.119794
0.625	−3.185460
0.875	−4.415121

Let's assume $f(t) = \rho_0 h_0 + \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3 = \rho_m = \mathrm{H}^{-1} f(t)$

So then,

$$f(t) = [0.330075, 0.755206, 0.939540, 0.959879]^T \quad (34)$$

We now solve

$$\rho_4 = \mathrm{H}^{-1} p(t) \quad (35)$$

$$\rho_4 = \mathrm{H}^{-1} f(t) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1.294925 \\ -2.119794 \\ -3.185460 \\ -4.415121 \end{bmatrix} \quad (36)$$

$$f_4 = [-2.446410, 0.739050, -2.001343, 2.413779]^T \quad (37)$$

Step 3

Approximate P^1, P^2

Using the simplified block pulse formulas;

$$P^1 = \begin{bmatrix} \frac{1}{2} & \frac{-1}{4} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{4} & 0 & \frac{-1}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{16} & 0 & 0 \\ \frac{1}{16} & \frac{-1}{16} & 0 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0.172 & -0.125 & -0.031 & -0.094 \\ 0.123 & -0.078 & 0.03125 & -0.031 \\ 0.047 & -0.0156 & -0.016 & 0 \\ 0.016 & -0.0156 & 0 & -0.016 \end{bmatrix} \quad (38)$$

Step 4

Substitute all the computed variables in equation (34)

$$C_4^T = [C_0, C_1, C_2, C_3] \quad (39)$$

Substituting equation (35), (36) and (37) into equation (38)

$$C_4 = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -1.176654 \\ 0.009153 \\ 0.091200 \\ -0.090600 \end{bmatrix} \quad (40)$$

substitute the value of C_4, H_m and p^2 into equation (29) and that gives,

$$\sigma(t) = C_m^T P_{m\times m}^2 H_m(t) + t = \begin{bmatrix} 0.108183 \\ 0.288064 \\ 0.391850 \\ 0.418968 \end{bmatrix} \quad (41)$$

Table 1: Numerical Solution and absolute error for example 1

t	*Exact solution*	*Haar solution* $m = 4$	*Absolute error*
0.125	0.116865	0.108183	0.008682
0.375	0.296525	0.288064	0.008462
0.625	0.396496	0.391850	0.004646
0.875	0.417526	0.418968	0.001441

Example 2

Consider a nonlinear ordinary differential equation.

$$\sigma''(t) + \sigma(t)\sigma'(t) + \sigma(t) = \sin(t)\cos(t) \quad (42)$$

$$\sigma(0) = 0, \sigma'(0) = 1 \quad (43)$$

Exact solution $\sigma(t) = sint$ (44)

Assume that,

$$\sigma''(t) = C_m H_m(t) \quad (45)$$

Integrate equation (45). It gives

$$\sigma'(t) = C_m P_{m\times m} H_m(t) + 1 \quad (46)$$

Integrate equation (46). It gives

$$\sigma(t) = C_m P_{m\times m}^2 H_m + t + \sigma(0) \quad (47)$$

Substitute equation (45), (46) and (47) into equation (42). It gives

$$C_m H_m(t) + C_m P_{m\times m}^2 H_m + t[C_m P_{m\times m} H_m(t) + 1] + C_m P_{m\times m}^2 H_m + t = \sin(t)\cos(t) \quad (48)$$

Equation 48 is the nonlinear algebraic system for the unknown Haar coefficients C

Equation (48) can be written as

$$C_m H_m(t_j) + [C_m P_{m\times m}^2 H_m + t_j][C_m P_{m\times m} H_m(t_j) + 1] + C_m P_{m\times m}^2 H_m + t_j = \sin(t_j)\cos(t_j) \quad (49)$$

$j = 1,2,3,4$

Table 2: Values of function $f(t)$ for varying values of t

x	$f(x)$
0.125	0.12370
0.375	0.34089
0.625	0.47449
0.875	0.49199

We will now build Haar wavelet matrix H for $m = 4$, this is already done in example 1

$$H_{4\times4} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (50)$$

$$H^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{2} \end{bmatrix} \quad (51)$$

$$p = \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} \quad (52)$$

$$p^2 = \begin{bmatrix} \frac{1}{64} & 0 & 0 & 0 \\ \frac{1}{16} & \frac{1}{64} & 0 & 0 \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{64} & 0 \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{64} \end{bmatrix} \quad (53)$$

Table 3: Values of function $f(t)$ for varying values of t.

t_j	$f(t_j)$
0.125	0.12370
0.375	0.34082
0.625	0.47449
0.875	0.49199

we now substitute equation (50), (51), (52) and (53) into equation (48) which gives

$$C_4 = \begin{bmatrix} -0.060760 \\ -0.173420 \\ -0.283650 \\ -0.372060 \end{bmatrix} \quad (54)$$

now substitute equation (50), (53) and (54) into equation (47), this gives

$$\sigma(t) = \begin{bmatrix} -0.060760 \\ -0.173420 \\ -0.283650 \\ -0.372060 \end{bmatrix} \begin{bmatrix} \frac{1}{64} & 0 & 0 & 0 \\ \frac{1}{16} & \frac{1}{64} & 0 & 0 \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{64} & 0 \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{64} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0.124675 \\ 0.366273 \\ 0.585097 \\ 0.767544 \end{bmatrix} \quad (55)$$

which is the desired result.

Table 4: Numerical Solution and absolute error for example 2

t	*Exact solution*	*Haar solution* $m = 4$	*Absolute error*
0.125	0.124675	0.124675	10^{-6}
0.375	0.366273	0.366273	10^{-6}
0.625	0.585097	0.585097	10^{-6}
0.875	0.767544	0.767544	10^{-6}

Example 3

Consider the ordinary differential equation with variable coefficient.

$$\sigma'' + (1+t)\sigma' + (2+t)\sigma = f(t) \quad (56)$$

Subject to $\sigma(0) = 1, \sigma'(0) = -1$ where $f(t) = 2e^{-t}$ (57)

Exact solution $\sigma = e^{-t}$

Assume that,

$$\sigma''(t) = C_m^T H_m(t) \quad (58)$$

Integrate equation (64) with respect to t using the stated conditions it gives,

$$\sigma'(t) - \sigma'(0) = C_m^T P_{m\times m}^1 H_m(t) \quad (59)$$

$$\sigma'(t) + 1 = C_m^T P_{m\times m}^1 H_m(t) \quad (60)$$

$$\sigma'(t) = C_m^T P_{m\times m}^1 H_m(t) - 1 \quad (61)$$

Integrate equation (61) with respect to t gives,

$$\sigma(t) - \sigma(0) = C_m^T P_{m\times m}^2 H_m(t) - t \quad (62)$$

$$\sigma(t) = C_m^T P_{m\times m}^2 H_m(t) - t + 1 \quad (63)$$

$f(t)$ expanded by Haar function as

$$f(t) = f_m^T H_m(t) \quad (64)$$

where $f_m^T(t)$ is constant vector vector.

substitute equation (58), (59), (60) and (61) into (56), we get

$$C_m^T H_m(t) + (1+t)[C_m^T P_{m\times m}^1 H_m(t) + (2+t) C_m^T P_{m\times m}^2 H_m(t)] = 2e^{-t} \quad (65)$$

$$C_m^T [H_m(t) + (1+t) P_{m\times m}^1 H_m(t) + (2+t)[P_{m\times m}^2 H_m(t)] = 2e^{-t} \quad (66)$$

Equation (66) is the algebraic form of equation (56).

We will now solve equation (63) manually using $m = 4$ by applying the following steps.

Construct the Haar matrix $H_{4\times 4}$

Define the function and compute the integrals

Build operational matrices p^1 and p^2

Compute f_m vector for $f(t) = 2e^{-t}$

Substitute the above stated result into equation (66)

Solve for C_m

$$H_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (67)$$

Simplifying the right hand side of equation (62) gives,

$$f(t) = 2e^{-t} + t^2 + 2t - 1 \quad (68)$$

$$f(0.125) = 1.030619 \quad (69)$$

$$f(0.375) = 1.265204 \quad (70)$$

$$f(0.625) = 1.711148 \quad (71)$$

$$f(0.875) = 2.349349 \quad (72)$$

hence,

$$p^1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{-1}{8} & \frac{-1}{8} \\ \frac{1}{4} & 0 & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{16} & 0 & 0 \\ \frac{1}{16} & -\frac{1}{16} & 0 & 0 \end{bmatrix}, \quad p^2 = \begin{bmatrix} \frac{11}{64} & -\frac{1}{8} & \frac{-1}{32} & \frac{-3}{32} \\ \frac{1}{8} & -\frac{3}{64} & -\frac{1}{32} & \frac{-1}{32} \\ \frac{3}{64} & -\frac{1}{64} & -\frac{1}{64} & 0 \\ \frac{1}{64} & -\frac{1}{64} & 0 & -\frac{1}{64} \end{bmatrix}, \quad (73)$$

$$f(t) = \begin{bmatrix} 1.030619 \\ 1.265204 \\ 1.711148 \\ 2.349349 \end{bmatrix} \quad (74)$$

substitute equation (67), (72), (73) and (74) into equation (66) to compute the coefficient matrix C_m^T gives,

$$C_m = \begin{bmatrix} 0.595825 \\ -0.016497 \\ 1.315630 \\ -1.024025 \end{bmatrix} \quad (75)$$

This yield the solution to equation (62)

$$\sigma = \begin{bmatrix} 0.882497 \\ 0.687289 \\ 0.535261 \\ 0.416862 \end{bmatrix}$$

Table 5: Numerical Solution and absolute error for example 3

t	$Exact\ solution$	$Haar\ solution\ m = 4$	$Absolute\ error$
0.125	0.882497	0.882497	0.017230
0.375	0.681289	0.687289	0.083984
0.625	0.535261	0.535261	0.063499
0.875	0.416862	0.416862	0.291180

Discussion

The results obtained from the three numerical examples shows the accuracy and efficiency of the Haar wavelet operational matrix method for solving second order differential equations. In each example, the second order differential equation was transformed into a system of algebraic equations using the Haar wavelet operational matrix. The computed Haar coefficients were then used to construct the approximate solutions, which showed excellent agreement with the corresponding exact solutions. This confirms that the proposed method provides accurate numerical approximations.

In summary, the numerical examples demonstrate that the Haar wavelet operational matrix method possesses several desirable computational features, including simplicity of implementation, reduced computational effort and excellent approximation accuracy. The orthogonality and compact support of the Haar basis functions contribute significantly to the efficiency of the algorithm by reducing the original differential equations to solvable algebraic systems. The results obtained are consistent with existing studies in the literature and demonstrate that the proposed method is a reliable numerical technique for solving second order differential equations. These findings suggest that the Haar wavelet operational matrix method can be extended to more challenging problems, including nonlinear, higher-dimensional, and fractional differential equations.

CONCLUSION

In this paper, we used the Haar wavelet operational matrix to solve the problems of second order differential equations. The aim is to show that the method is efficient and accurate by using the proposed method to transform the differential equation into a system of algebraic equations which we then solved to get the desired results. This paper proved that the Haar wavelet operational matrix is not only efficient but also accurate and more convenient.

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