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The Generalized Exponentiated Transmuted Weibull Distribution: Properties and Applications

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ABSTRACT

Lifetime and survival data occur frequently in engineering, biomedical, and actuarial studies. Accurate modeling of such data is crucial for understanding failure mechanisms, predicting system reliability, and guiding decision making. This paper introduces a new flexible lifetime distribution called the Generalized Exponentiated Transmuted Weibull (GETW) distribution. The proposed distribution integrates generalization, exponentiation, and transmutation to enhance flexibility, capturing various hazards, including increasing, decreasing, unimodal, and bathtub-shaped hazards. Statistical properties, parameter estimation via maximum likelihood, a Monte Carlo simulation study, and real data applications are presented.

Keywords: Exponentiated Wiebull, Transmuted Weibull, Weibull Distribution, Properties, Application

INTRODUCTION

Survival analysis has emerged as a crucial statistical methodology in diverse fields, from medical research and epidemiology to engineering and economics. At its core, this analytical approach focuses on modeling and analyzing time-to-event data, where the primary interest lies in understanding the time until a specific event occurs Kleinbaum and Klein (1996) The development of sophisticated survival regression models has become increasingly important as researchers face more complex datasets and demanding analytical requirements.

The Weibull distribution, introduced by, (Weibull, 1951) has long served as a fundamental tool in survival analysis due to its flexibility in modeling various hazard patterns. However, as datasets become more complex and research questions more nuanced, traditional Weibull models often prove inadequate in capturing the intricate patterns present in modern survival data. This limitation has sparked interest in developing more flexible and robust modifications of the Weibull distribution.

Two significant modifications of the Weibull distribution have shown particular promise: the exponentiated Weibull distribution (EWD) and the transmuted Weibull distribution (TWD). The EWD, introduced by Mudholkar and Srivastava in 1993, offers enhanced flexibility in modeling both monotonic and non-monotonic hazard rates. Similarly, the TWD, developed through quadratic rank transmutation maps, provides additional capability in modeling complex survival patterns that deviate from standard Weibull assumptions (Shaw and Buckley, 2007)

Despite these advances, there remains a significant gap in the integration of these modifications into comprehensive survival regression frameworks. Current models often fail to fully exploit the advantageous properties of both distributions simultaneously, leading to potential limitations in their practical applications. Furthermore, the increasing complexity of survival data in various fields demands more sophisticated modeling approaches that can effectively handle multiple predictor variables while maintaining computational efficiency. (Wang et al., 2019).

Motivation and Background

The Weibull distribution is one of the most widely used lifetime models in survival analysis and reliability engineering due to its ability to model increasing and decreasing hazard rate functions. Despite its popularity, real-life survival data from biomedical and engineering applications often exhibit more complex characteristics such as non"monotonic hazard rates, heavy tails, early failures, long"term survivors, and varying degrees of skewness. In such situations, the classical Weibull distribution and many of its existing extensions may provide inadequate fits. To overcome these limitations, several generalisation techniques have been introduced in the literature, among which the exponentiation and transmutation methods have received considerable attention. The exponentiation technique introduces an additional shape parameter that enhances flexibility in modelling tail behaviour and skewness. the transmutation approach modifies the baseline distribution through a quadratic rank transmutation map, thereby improving the ability of the model to capture diverse hazard rate patterns. Although each of these approaches individually improves modelling flexibility, combining them within a unified framework provides a more powerful lifetime distribution.

The development of the Generalized Exponentiated Transmuted Weibull (GETW) distribution is therefore motivated by the need for a highly flexible distribution capable of modelling a wide range of survival and reliability phenomena within a single framework. By incorporating multiple shape parameters, the proposed model is able to accommodate increasing, decreasing, bathtub"shaped, and unimodal hazard rate functions, making it particularly suitable for complex real"world data encountered in medical survival studies and industrial reliability analysis

Theoretical Foundation

Weibull Distribution Continuous probability distribution widely used in survival analysis and reliability engineering. It is characterized by its flexibility in modeling different types of hazard rates (increasing, decreasing, or constant). The Weibull distribution has been fundamental in survival analysis because of its flexibility in modeling various hazard rates. The standard Weibull distribution's probability density function is characterized by shape and scale parameters, making it suitable for modeling time-to-event data. However, limitations in handling complex survival patterns led to the development of modified versions.

MATERIALS AND METHODS

Methods

Baseline Weibull Distribution

The Weibull distribution is one of the most widely used lifetime models in survival and reliability studies. Introduced by Waloddi Weibull in 1951, it is defined by two parameters (scale and shape) and is particularly suited for modeling monotonic hazard rates, either increasing or decreasing.

Exponentiated Weibull Distribution

The exponentiated Weibull distribution, introduced by (Mudholkar and Srivastava, 1993), adds an additional shape parameter to the standard Weibull distribution. This modification allows for greater flexibility in modeling non-monotonic hazard rates, addressing a key limitation of the standard Weibull model. The EW model provides increased flexibility to accommodate a wider variety of hazard shapes.

Transmuted Weibull Distribution

The transmuted Weibull distribution, developed through the quadratic rank transmutation map (QRTM), introduces a transmutation parameter that allows for greater shape flexibility. This modification has proven particularly useful in handling heterogeneous populations.

The integration of novel regression models, particularly the exponentiated and transmuted Weibull distributions, has garnered attention in survival analysis due to their increased flexibility in modeling survival data. Traditional models, such as the exponential and standard Weibull distributions, often fall short when dealing with complex hazard functions observed in real-world data. The exploration of these novel distributions is evidenced in several studies. For instance, the recent work by (Klakattawi, 2022) introduced an extended Weibull distribution aimed explicitly at cancer patient survival analysis, demonstrating how the incorporation of additional parameters can significantly enhance model fit. Using a dataset of cancer patients, the study illustrated the inadequacy of classical distributions compared to its proposed model, establishing the potential value of expanded parameterization in survival modeling.

Moreover, (Swihart and Bandyopadhyay, 2021) showed the importance of bridging different types of survival models. Their approach reduces computational burdens while maintaining the integrity of the relationships between various survival estimates. The implications of their work are pertinent for high-dimensional databases typical in health research today.

Theoretical Framework

Understanding Weibull Distribution (Base Model)

The Weibull distribution is widely used in survival analysis, reliability engineering, and failure time analysis because it can model increasing, decreasing, or constant hazard rates.

Weibull Probability Density Function (PDF):

$$f_W(x;\theta,k)=\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k},\quad x>0 \tag{1}$$

where: - $\theta > 0$ (scale parameter) - $k > 0$ (shape parameter)

Weibull Cumulative Distribution Function (CDF):

$$F_W(x)=1-e^{-(\lambda x)^k}$$

This gives the probability that a failure (or event) occurs by time *x*.

Exponentiated-G (Exp-G) Distribution

Let $G(x)$ and $g(x)$ be the cumulative distribution function (CDF) and probability density function (PDF), respectively, of a baseline distribution. Let $\alpha > 0$ be a shape (exponentiation) parameter.

Then the exponentiated-G distribution is defined as follows:

Cumulative Distribution Function (CDF)

$$F(x)=[G(x)]^{\alpha}$$

Probability Density Function (PDF)

$$f(x)=\alpha\, g(x)[G(x)]^{\alpha-1}$$

Transmuted Weibull PDF

$$f_{TW}(x)=\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k}\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right] \tag{5}$$

Transmuted Weibull CDF

$$F_{TW}(x)=(1-\lambda)\left(1-e^{-(x/\theta)^k}\right)+\lambda\left(1-e^{-(x/\theta)^k}\right)^2 \tag{6}$$

Model Development and Methodology

This section highlight the development of the Generalized Exponentiated Transmuted Weibull (GETW) distribution for survival data with applications in biomedical and reliability studies. The section outlines the probability structure and estimation techniques.

From Weibull to Generalized Exponentiated Transmuted Weibull

Step 1 Baseline Weibull Distribution

Let *X* follow a Weibull distribution with scale parameter $\theta > 0$ and shape parameter $k > 0$. Its CDF and PDF are

$$F_W(x;\theta,k)=1-\exp\left[-(x/\theta)^k\right],\quad x\ge 0, \tag{7}$$

$$f_W(x;\theta,k)=\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\exp\left[-(x/\theta)^k\right]. \tag{8}$$

Step 2 The transmutation parameter λ ($-1<\lambda<1$) applied to the Weibull CDF:

$$T(x)=(1-\lambda)F_W(x)+\lambda\left(F_W(x)\right)^2. \tag{9}$$

Differentiating:

$$T'(x)=f_W(x)\left[1+\lambda-2\lambda\exp(-(x/\theta)^k)\right]. \tag{10}$$

Step 3 Exponentiation (inner) Introduce parameter $\alpha > 0$:

$$U(x)=\left(T(x)\right)^{\alpha}$$

Step 4 Generalization (outer) Introduce $\beta > 0$ and define the GETW cumulative distribution:

$$F_{GETW}(x)=1-\left(1-U(x)\right)^{\beta}=1-\left\{1-\left(T(x)\right)^{\alpha}\right\}^{\beta}. \tag{11}$$

Step 5 GETW PDF [PDF of the GETW Distribution] The PDF of the GETW distribution is

$$f_{GETW}(x;\alpha,\beta,\lambda,\theta,k)=\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k}\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right]\times\left\{(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right\}^{\alpha-1}\times\left\{1-\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta-1}. \tag{12}$$

Probability Density Function (PDF) Plot

The probability density function (PDF) in Figure 1 shows how the probability mass is distributed over time. Depending on the parameter choices, the PDF showed the following.

- Unimodal shapes (single peak),

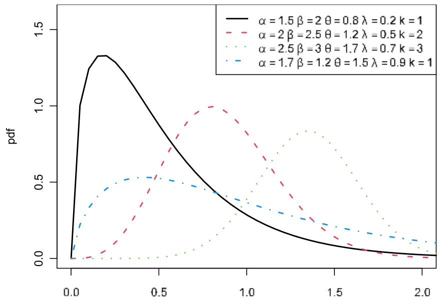

Figure 1: Plot of PDF of the GETW Distribution for Different Parameter Values

Skewed distributions (right- or left-skew), Heavy-tailed behavior for larger scale/shape values.

The GETW can model lifetimes with early failures, long tails, or moderate spread, making it flexible for biomedical and reliability data.

The cumulative distribution function (CDF) is

$$F(x)=1-\left\{1-\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta}, \quad (13)$$

Cumulative Distribution Function (CDF) GETW

The cumulative distribution function (CDF) in Figure 2 represents the accumulated probability up to time x. The plots showed:

Steep CDFs (rapid rise) ⇒ short lifetimes / higher early risk,

Flat/slower CDFs ⇒ longer survival times,

Changes mainly driven by the scale parameter (θ), while shape (α, β) and transmutation (λ) adjust the curvature.

Interpretation

The GETW CDF flexibly captures populations ranging from early failure-prone to long-lived systems/patients.

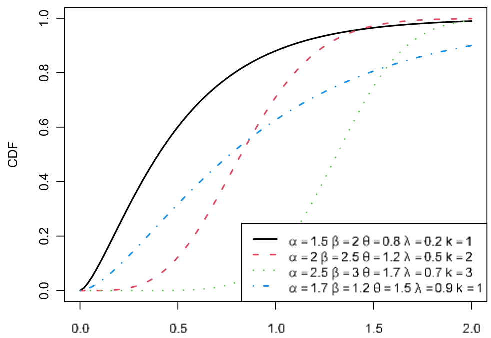

Figure 2: Plot of CDF of the GETW Distribution for Different Parameter Values

Validity Test of the PDF of the GETW DISTRIBUTION

The function $f_{GETW}(x)$ defined in equation 12 is a valid probability density function on $[0, \infty)$.

Proof. (Non-Negativity)

For $x \geq 0$, we have $T(x) \in [0, 1]$. Each factor in

Begin $f_{GETW}(x)=\alpha\beta T(x)^{\alpha-1}(1-T(x)^{\alpha})^{\beta-1}T'(x)$

is nonnegative under $\alpha, \beta > 0$, $|\lambda| < 1$, so $f_{GETW}(x) \geq 0$.

(Normalization). Compute

$$\int_0^{\infty} f_{GETW}(x)\,dx=\alpha\beta\int_0^{\infty} T(x)^{\alpha-1}\left(1-T(x)^{\alpha}\right)^{\beta-1}T'(x)\,dx.$$

Let $v = T(x)$. Since $T(0) = 0$, $\lim_{x\to\infty} T(x) = 1$, and $dv = T'(x)\,dx$, the integral becomes

$\alpha\beta\int_0^1 v^{\alpha-1}(1-v^{\alpha})^{\beta-1}\,dv$. Substitute $t = v^{\alpha}$, $dt = {}_{\alpha}v^{\alpha-1}dv$, giving

$\beta\int_0^1 (1-t)^{\beta-1}\,dt=1.$

Hence $\int_0^{\infty} f_{GETW}(x)\,dx = 1$.

A probability density function (PDF) $f(x)$ is valid if it satisfies two conditions (Kolassa, 2006) (i) non-negativity: $f(x) \geq 0$ for all x, and (ii) normalization: $\int_0^{\infty} f(x)\,dx = 1$.

Step 1: The GETW PDF

The PDF of the Generalized Exponentiated Transmuted Weibull (GETW) distribution is given by

$$f_{GETW}(x;\alpha,\beta,\lambda,\theta,k)=\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k}\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right]\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha-1}\left\{1-\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta-1}.$$

Step 2: Non-Negativity

The term $\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k}$ is nonnegative for $x > 0$.

- The component $(1-e^{-(x/\theta)^k}) \in (0,1)$, hence its powers are nonnegative.

- The factor $\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right] \geq 0 \, when -1<\lambda<1.$

Therefore, $f_{GETW}(x) \geq 0$ for all $x > 0$.

Step 3: Normalization

We now prove our point.

$\int_0^\infty f_{GETW}(x)\,dx = 1.$ Define the transformation

$$u=(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2, \quad u\in[0,1].$$

The derivative of u w.r.t. x is

$$\frac{du}{dx}=\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} e^{-(x/\theta)^k}\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right].$$

Thus, the PDF can be rewritten as

$$f_{GETW}(x)=\alpha\beta u^{\alpha-1}(1-u^{\alpha})^{\beta-1}\cdot\frac{du}{dx}.$$

Hence, $\int_0^\infty f_{GETW}(x)\,dx=\int_0^1 \alpha\beta u^{\alpha-1}(1-u^{\alpha})^{\beta-1}\,du.$

Now let $t=u^{\alpha} \Rightarrow dt=\alpha u^{\alpha-1}du.$ The integral becomes

$$\int_0^1 \alpha\beta u^{\alpha-1}(1-u^{\alpha})^{\beta-1}\,du=\int_0^1 \beta(1-t)^{\beta-1}dt.$$

Evaluating:

$$\int_0^1 \beta(1-t)^{\beta-1}dt=\left[-(1-t)^{\beta}\right]_0^1=1.$$

Step 4: Conclusion

The GETW PDF is nonnegative and integrates to unity. Therefore,

$f_{GETW}(x)$ is a valid probability density function.

Statistical Properties

We frequently begin with a complex probability density function (PDF) or a cumulative distribution function (CDF). When it comes to deriving properties like moments, quantiles, entropy, or order statistics, the original form can be too cumbersome for integration, differentiation, or inversion. Therefore, we typically simplify or reformat the PDF and CDF before proceeding with our analysis.

Final reduced CDF

$$F(x)=\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\sum_{t=0}^{\infty}\Phi_{m,j,t}\,e^{-(j+t)z},$$

Where $\Phi_{m,j,t}=(-1)^{m+1+t}\binom{\beta}{m}\binom{\alpha m}{j}\binom{\alpha m}{t}\lambda^{j}.$

the PDF series is

$$f(x)=\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\sum_{r=1}^{\infty}\gamma_r\,y^r=\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\sum_{r=1}^{\infty}\gamma_r\,e^{-r(x/\theta)^k}. \quad (17)$$

Formally, let $X \sim GETW(\alpha,\beta,\lambda,\theta,k),$

Survival Function

The survival function plays a central role in survival analysis and reliability theory, as it represents the probability that a system or individual survives beyond a specified time x. For the Generalized Exponentiated Transmuted Weibull (GETW) distribution, the survival function is obtained directly from the cumulative distribution function (CDF) by taking its complement, $S(x) = 1 - F(x)$. The explicit form of the survival function is given by:

$$S(x;\alpha,\beta,\theta,\lambda,k)=\left\{1-\left[(1-\lambda)\left(1-e^{-(x/\theta)^k}\right)+\lambda\left(1-e^{-(x/\theta)^k}\right)^2\right]^{\alpha}\right\}^{\beta}, \quad x>0. \quad (16)$$

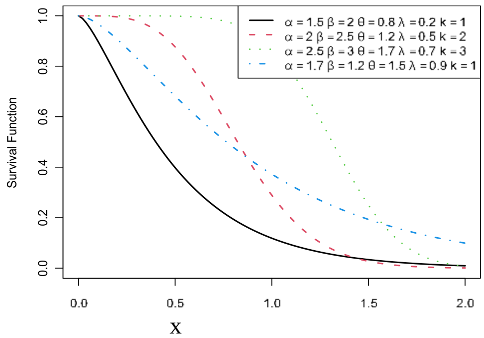

Figure 3: Plot of Survival Function of the GETW Distribution for Different Parameter Values

Figure 3 shows that GETW survival functions adapt to early mortality, long-term stability, and aging effects, making them relevant both for patient survival and component reliability.

Hazard Function

The hazard function is formally defined as

$$h(x)=\frac{f(x)}{S(x)},$$

where $f(x)$ denotes the probability density function (PDF) and $S(x)$ represents the survival function. By substituting the expressions of the GETW density and survival function into this definition, we obtain:

$$h(x;\alpha,\beta,\theta,\lambda,k)=\frac{\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k}\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right]\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha-1}}{\left\{1-\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta}}.$$

Hazard Function for GETW Distribution

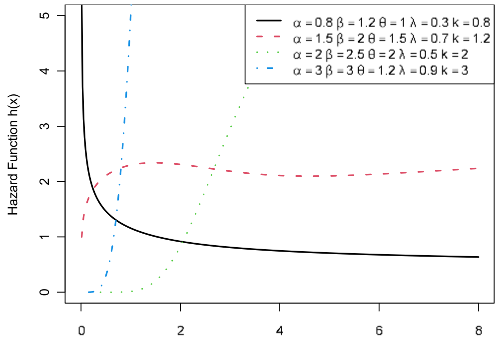

Figure 4: Plot of Hazard Function of the GETW Distribution for Different Parameter Values

Hazard Function (HF) Plot

The hazard function h(x) = f(x)/(x) gives the instantaneous failure rate. The Plot in figure 4 demonstrated a wide variety of hazard shapes:

Moment

We aim to evaluate the *p*-th moment:

$$\mu_p = \sum_{r=1}^{\infty} \frac{\alpha\beta k\gamma_r}{\theta^k} \int_0^{\infty} x^{p+k-1} e^{-r(x/\theta)^k}\, dx$$

Step 1: Evaluate the Integral

Let us use the substitution:

$$u = r\left(\frac{x}{\theta}\right)^k \Rightarrow x = \theta\left(\frac{u}{r}\right)^{1/k}, \quad dx = \frac{\theta}{kr}\left(\frac{u}{r}\right)^{(1-k)/k} du$$

Then, $x^{p+k-1} = \theta^{p+k-1}\left(\frac{u}{r}\right)^{(p+k-1)/k}$

Substituting into the integral:

$$\int_0^{\infty} x^{p+k-1} e^{-r(x/\theta)^k}\, dx = \frac{\theta^{p+k}}{kr^{1+p/k}} \Gamma\left(1+\frac{p}{k}\right)$$

Step 2: Substitute into the Summation

$$\mu_p = \sum_{r=1}^{\infty} \frac{\alpha\beta k\gamma_r}{\theta^k} \cdot \frac{\theta^{p+k}}{kr^{1+p/k}} \Gamma\left(1+\frac{p}{k}\right)$$

Simplify:

$$\mu_p = \alpha\beta\theta^p \Gamma\left(1+\frac{p}{k}\right) \sum_{r=1}^{\infty} \frac{\gamma_r}{r^{1+p/k}}$$

Final Result:

$$\mu_p = \alpha\beta\theta^p \Gamma\left(1+\frac{p}{k}\right) \sum_{r=1}^{\infty} \frac{\gamma_r}{r^{1+p/k}}$$

Given the general moment formula:

$$\mu_p = \alpha\beta\theta^p \Gamma\left(1+\frac{p}{k}\right) \sum_{r=1}^{\infty} \frac{\gamma_r}{r^{1+p/k}}$$

Moment Generating Function

Step 1: Expand the exponential

$$e^{tx} = \sum_{p=0}^{\infty} \frac{t^p x^p}{p!}$$

substitute into the integral:

$$\int_0^{\infty} e^{tx-r(x/\theta)^k}\, dx = \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_0^{\infty} x^p e^{-r(x/\theta)^k}\, dx$$

Step 2: Evaluate the Inner Integral

$$\int_0^{\infty} x^p e^{-r(x/\theta)^k}\, dx = \frac{\theta^{p+1}}{kr^{(p+1)/k}} \Gamma\left(\frac{p+1}{k}\right)$$

Step 3: Substitute Back

$$M_X(t) = \sum_{r=1}^{\infty} \frac{\alpha\beta k\gamma_r}{\theta^k} \sum_{p=0}^{\infty} \frac{t^p}{p!} \cdot \frac{\theta^{p+1}}{kr^{(p+1)/k}} \Gamma\left(\frac{p+1}{k}\right)$$

simplify constants:

$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} \cdot \alpha\beta\theta^{p+1} \Gamma\left(\frac{p+1}{k}\right) \sum_{r=1}^{\infty} \frac{\gamma_r}{r^{(p+1)/k}}$$

$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} \cdot \alpha\beta\theta^{p+1} \Gamma\left(\frac{p+1}{k}\right) \sum_{r=1}^{\infty} \frac{\gamma_r}{r^{(p+1)/k}}$$

Quantile Function

Given the cumulative distribution function

$$F(x) = 1 - \left\{1 - \left[(1-\lambda)(1-e^{-(x/\theta)^k}) + \lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta}$$

Let $u = F(x)$. Then:

$$1-u = \left\{1 - \left[(1-\lambda)(1-e^{-(x/\theta)^k}) + \lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta}$$

Raise both sides to the power $1/\beta$:

$$(1-u)^{1/\beta} = 1 - \left[(1-\lambda)(1-e^{-(x/\theta)^k}) + \lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}$$

Rearranging:

$$\left[(1-\lambda)(1-e^{-(x/\theta)^k}) + \lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha} = 1-(1-u)^{1/\beta}$$

Raise both sides to the power $1/\alpha$:

$$(1-\lambda)(1-e^{-(x/\theta)^k}) + \lambda(1-e^{-(x/\theta)^k})^2 = \left[1-(1-u)^{1/\beta}\right]^{1/\alpha}$$

Let:

$$z = 1-e^{-(x/\theta)^k}$$

Then:

$$(1-\lambda)z + \lambda z^2 = \left[1-(1-u)^{1/\beta}\right]^{1/\alpha}$$

This is a quadratic in *z*:

$$\lambda z^2 + (1-\lambda)z - \left[1-(1-u)^{1/\beta}\right]^{1/\alpha} = 0$$

Solving for *z* using the quadratic formula:

$$z=\frac{-(1-\lambda)+\sqrt{(1-\lambda)^2+4\lambda\left[1-(1-u)^{1/\beta}\right]^{1/\alpha}}}{2\lambda}$$

Now recall:

$$z=1-e^{-(x/\theta)^k}\Rightarrow e^{-(x/\theta)^k}=1-z$$

Taking logarithms:

$$-\left(\frac{x}{\theta}\right)^k=\ln(1-z)\Rightarrow x=\theta\left[-\ln(1-z)\right]^{1/k}$$

Final Quantile Function

$$Q(u)=\theta\left[-\ln\left(1-\frac{-(1-\lambda)+\sqrt{(1-\lambda)^2+4\lambda\left[1-(1-u)^{1/\beta}\right]^{1/\alpha}}}{2\lambda}\right)\right]^{1/k}$$

Order Statistics

Given:

$$F(x)=\sum_{n=1}^{\infty}a_n e^{-n(x/\theta)^k}$$

$$f(x)=\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\sum_{r=1}^{\infty}\gamma_r e^{-r(x/\theta)^k},\qquad \gamma_r=(1+\lambda)C_{r-1}-2\lambda C_{r-2},\quad C_q=\sum_{m=0}^{q}\phi_m\chi_{q-m}$$

The PDF of the i-th order statistic from a sample of size n is:

$$f_{X_{(i)}}(x)=\frac{n!}{(i-1)!(n-i)!}\left[\sum_{n=1}^{\infty}a_n e^{-n(x/\theta)^k}\right]^{i-1}\left[1-\sum_{n=1}^{\infty}a_n e^{-n(x/\theta)^k}\right]^{n-i}\cdot\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\sum_{r=1}^{\infty}\gamma_r e^{-r(x/\theta)^k}$$

Substituting the expressions:

$$f_{X_{(i)}}(x)=\frac{n!}{(i-1)!(n-i)!}\left[\sum_{n=1}^{\infty}a_n e^{-n(x/\theta)^k}\right]^{i-1}\left[1-\sum_{n=1}^{\infty}a_n e^{-n(x/\theta)^k}\right]^{n-i}\cdot\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\sum_{r=1}^{\infty}\gamma_r e^{-r(x/\theta)^k}$$

Parameter Estimation

Maximum Likelihood Estimation for GETW Distribution

Given the probability density function:

$$f_{GETW}(x;\alpha,\beta,\lambda,\theta,k)=\alpha\beta\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-(x/\theta)^k}\left[1+\lambda-2\lambda e^{-(x/\theta)^k}\right]\times\left\{(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right\}^{\alpha-1}\times\left\{1-\left[(1-\lambda)(1-e^{-(x/\theta)^k})+\lambda(1-e^{-(x/\theta)^k})^2\right]^{\alpha}\right\}^{\beta-1}$$

Let $x_1, x_2, \ldots x_n$ be a random sample from this distribution. The log-likelihood function is:

$$\ell(\alpha,\beta,\lambda,\theta,k)=\sum_{i=1}^{n}\log f_{GETW}(x_i;\alpha,\beta,\lambda,\theta,k)$$

$$=\sum_{i=1}^{n}\left[\log\alpha+\log\beta+\log k-\log\theta+(k-1)\log\left(\frac{x_i}{\theta}\right)-\left(\frac{x_i}{\theta}\right)\right.$$
$$+\log\left(1+\lambda-2\lambda e^{-(x_i/\theta)^k}\right)$$
$$+(\alpha-1)\log\left((1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right)$$

k

$$+(\beta-1)\log\left(1-\left[(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right]^{a}\right.$$

Partial Derivatives

With respect to α:

$$\frac{\partial\ell}{\partial\alpha}=\sum_{i=1}^{n}\left[\frac{1}{\alpha}+\log\left((1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right)-\frac{\left[(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right]^{\alpha}\cdot\log\left((1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right)}{1-\left[(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right]^{\alpha}}\right]$$

With respect to β:

$$\frac{\partial\ell}{\partial\beta}=\sum_{i=1}^{n}\left[\frac{1}{\beta}+\log\left(1-\left[(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right]^{\alpha}\right)\right]$$

With respect to λ:

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\left[\frac{1-2e^{-(x_i/\theta)^k}}{1+\lambda-2\lambda e^{-(x_i/\theta)^k}}+(\alpha-1)\cdot\frac{-(1-e^{-(x_i/\theta)^k})+(1-e^{-(x_i/\theta)^k})^2}{(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2}-(\beta-1)\cdot\frac{\alpha\left[(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right]^{\alpha-1}\left[-(1-e^{-(x_i/\theta)^k})+(1-e^{-(x_i/\theta)^k})^2\right]}{1-\left[(1-\lambda)(1-e^{-(x_i/\theta)^k})+\lambda(1-e^{-(x_i/\theta)^k})^2\right]^{\alpha}}\right]$$

With respect to θ:

$$\frac{\partial\ell}{\partial\theta}=\frac{k}{\theta}\sum_{i=1}^{n}[\left(\frac{x_i}{\theta}\right)^k-1-\frac{2\lambda\left(\frac{x_i}{\theta}\right)^k e^{-(x_i/\theta)^k}}{1+\lambda-2\lambda e^{-(x_i/\theta)^k}}-(\alpha-1)\frac{\left(\frac{x_i}{\theta}\right)^k e^{-(x_i/\theta)^k}\left[(1-\lambda)+2\lambda\left(1-e^{-(x_i/\theta)^k}\right)\right]}{(1-\lambda)\left(1-e^{-(x_i/\theta)^k}\right)+\lambda\left(1-e^{-(x_i/\theta)^k}\right)^2}][\quad+(\beta-1)\frac{\alpha\left(\frac{x_i}{\theta}\right)^k e^{-(x_i/\theta)^k}\left[(1-\lambda)+2\lambda\left(1-e^{-(x_i/\theta)^k}\right)\right]\left[(1-\lambda)\left(1-e^{-(x_i/\theta)^k}\right)+\lambda\left(1-e^{-(x_i/\theta)^k}\right)^2\right]^{\alpha-1}}{1-\left[(1-\lambda)\left(1-e^{-(x_i/\theta)^k}\right)+\lambda\left(1-e^{-(x_i/\theta)^k}\right)^2\right]^{\alpha}}].$$

With respect to k:

$$\frac{\partial\ell}{\partial k}=\sum_{i=1}^{n}[\frac{1}{k}+\ln\left(\frac{x_i}{\theta}\right)-t_i\ln\left(\frac{x_i}{\theta}\right)]][\quad+\sum_{i=1}^{n}\frac{2\lambda t_i\ln\left(\frac{x_i}{\theta}\right)u_i}{1+\lambda-2\lambda u_i}][\quad-(\alpha-1)\sum_{i=1}^{n}\frac{t_i\ln\left(\frac{x_i}{\theta}\right)u_i\left[(1-\lambda)+2\lambda(1-u_i)\right]}{(1-\lambda)(1-u_i)+\lambda(1-u_i)^2}][\quad-(\beta-1)\sum_{i=1}^{n}\frac{\alpha t_i\ln\left(\frac{x_i}{\theta}\right)u_i\left[(1-\lambda)+2\lambda(1-u_i)\right]\left[(1-\lambda)(1-u_i)+\lambda(1-u_i)^2\right]^{\alpha-1}}{1-\left[(1-\lambda)(1-u_i)+\lambda(1-u_i)^2\right]^{\alpha}}.$$

RESULTS AND DISCUSSION

Results

This section introduced the simulation study designed to examine the behaviour and consistency of maximum likelihood estimation, as well as the application of real data to observe the performance of the proposed model and other competing models.

Optimization Procedure

Parameter estimation was carried out using the BFGS optimization algorithm implemented in the optim() function in R. The iterative procedure was terminated when the absolute change in the log-likelihood function between successive iterations was less than 10^{-8}.

Simulation Study
In this section, a simulation study was carried out for the new model to assess the consistency of the estimated parameters obtained using MLE methods.

Simulation Study of GETW Distribution
To assess the performance of the proposed GETW distribution, a simulation study was conducted using the Monte Carlo Simulation method to compute the Estimates, bias and root mean square error (RMSE) of the estimated parameters from the maximum likelihood estimates. The simulated data are produced using the quantile function of the GETW distribution for different sample sizes: n= 30, 100,200,400,450, 500 and 600 replicated 1000 times. For each sample size $\alpha = 2.5$, $\beta = 1.3$, $\lambda = 0.6$, $\theta = 1.5$ and $k = 1.8$ Table 1 presents the results of the estimates (Est.), bias and root mean square error from the new distribution.

Table 1: Result of Simulation for Different Values of Parameters

n Value	Parameters	$\alpha = 2.5$	$\beta = 1.3$	$\lambda = 0.6$	$\theta = 1.5$	$k = 1.8$
n=30	Estimate	3.272678	1.887429	0.392410	1.600774	3.836431
	Bias	0.772678	0.587429	-0.207590	0.100774	2.036431
	RMSE	3.541195	3.324474	1.447617	1.129468	3.558087
n=100	Estimate	3.185389	1.364043	0.300759	1.396322	2.622439
	Bias	0.685389	0.064043	-0.299241	-0.103678	0.822439
	RMSE	3.079244	2.303652	1.510175	0.870832	1.884743
n=200	Estimate	3.118968	1.310175	0.251562	1.364811	2.313960
	Bias	0.618968	0.010175	-0.348438	-0.135189	0.513960
	RMSE	2.670640	1.971818	1.520895	0.732420	1.333553
n=400	Estimate	3.116033	1.622261	0.241355	1.425099	2.197204
	Bias	0.616033	0.322261	-0.358645	-0.074901	0.397204
	RMSE	2.635362	2.171401	1.537633	0.681039	1.236368
n=450	Estimate	3.051784	1.731755	0.310525	1.447093	2.113657
	Bias	0.551784	0.431755	-0.289475	-0.052907	0.313657
	RMSE	2.528683	2.261938	1.445172	0.712865	1.134392
n=500	Estimate	3.055917	1.640609	0.283341	1.444169	2.119939
	Bias	0.555917	0.340609	-0.316659	-0.055831	0.319939
	RMSE	2.494437	2.170016	1.470058	0.682055	1.102204
n=600	Estimate	2.917787	1.732655	0.258738	1.482085	2.157640
	Bias	0.417787	0.432655	-0.341262	-0.017915	0.357640
	RMSE	2.259631	2.402541	1.515746	0.684426	1.134298

The results of the Monte Carlo Simulations are shown in table 1. The values of bias and RMSEs tend to zero and the estimates tend to the true parameter values as the sample size increases, indicating that the estimates are consistent

Application to Real Data
Turbocharger Failure Time Data
This subsection presents the analysis of the time-to-failure data of a turbocharger from a particular type of engine. The dataset was originally reported by Xu et al. (2003) and consists of 40 observations. The observed failure times are given as follows
1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0.

Table 2: Parameter Estimates and Model Selection Criteria

Model	Parameter Estimates	−LL	AIC	BIC
5*GETW	$\alpha = 0.2646$	77.0119	164.0239	172.4683
	$\beta = 1.0142$			
	$\lambda = -0.8663$			
	$\theta = 10.1381$			
	$k = 0.4620$			
2*W	$\gamma = 3.8799$	82.4757	170.9515	176.0181
	$\lambda = 6.9168$			
3*GW	$\alpha = 1.1011$	112.1535	230.307	235.3736
	$\beta = 1.0000$			
	$\lambda = 0.2000$			
3*HLGW	$\lambda = 1.0000$	167.5867	341.1735	346.2401
	$\alpha = 0.2000$			
	$\theta = 1.0000$			
3*TW	$\alpha = 3.5887$	82.1863	170.3726	175.4392
	$\gamma = 6.5943$			
	$\lambda = -0.3362$			

Model	Parameter Estimates	−LL	AIC	BIC
3*EW	$\alpha = 0.7048$ $\theta = 3.7524$ $\lambda = 2.0487$	133.8744	273.7489	278.8155

Density Plot

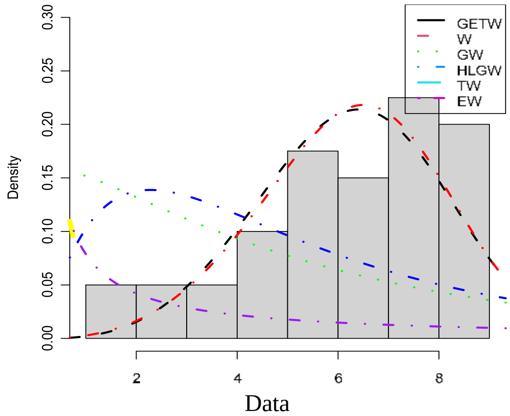

Figure 5: Density Plot

Model Fit Comparison
Model adequacy is evaluated using the log-likelihood, AIC and BIC. Lower values of these criteria indicate better model performance. The GETW model produced the smallest negative log-likelihood value ($-LL = 77.0119$), indicating the best fit to the observed data, the GETW model also yielded the lowest AIC value (164.0239), which shows the best trade-off between goodness-of-fit and model complexity, similarly, the BIC value for the GETW model (172.4683) is the smallest among all competing models, confirming its superiority even under a stronger penalty for model complexity. Based on the log-likelihood, AIC and BIC criteria, the Generalized Exponentiated Transmuted Weibull (GETW) model provides the best representation of the data. The results demonstrate that the additional shape and parameter significantly improve the flexibility and modelling capability of the distribution. The GETW model is preferred over the GW,HLGW, TW, EW and classical Weibull models for the analysed survival dataset.

Second Dataset
The data set is on chemotherapy for stage B-C colon cancer it is a censored data set. The data used in this study were obtained from the colon dataset in the R survival package. The dataset contains information from one of the first successful clinical trials of adjuvant chemotherapy for Stage B/C colon cancer and has been widely used in survival analysis research Moertel et al. (1990, 1991); Lin (1994).

Type of Censoring
The chemotherapy dataset for patients with stage B–C colon cancer is a right-censored survival dataset. In right-censored data, the exact survival time is not observed for some individuals because the event of interest has not occurred before the end of the study period or because subjects are lost to follow-up. Consequently, it is only known that the true survival time exceeds the observed censoring time. Right censoring is the most common form of censoring encountered in survival analysis and reliability studies Klein and Moeschberger (2003); Kleinbaum and Klein (2012).
The censoring indicator is defined as

$$\delta_i = \begin{cases} 1, & \text{if the event is observed,} \\ 0, & \text{if the observation is right-censored.} \end{cases}$$

The likelihood function for the censored observations is therefore constructed using both the probability density function and the survival function of the proposed GETW distribution.

Table 3: Parameter Estimates and Model Selection Criteria

Model	Parameter Estimates	*−LL*	AIC	BIC
5*GETW	$\alpha = 4.8136837$ $\beta = 0.4178606$ $\lambda = -0.8460234$ $\theta = 499.9930187$ $k = 0.4250641$	8218.032	16446.06	16473.70
4*GEW	$\alpha = 4.0108176$ $\beta = 3.0400609$ $\theta = 1537.5508111$ $k = 0.2996634$	8253.612	16515.22	16537.33
3*EW	$\alpha = 1.9911037$ $\theta = 1537.5578113$ $k = 0.5108477$	8261.236	16528.47	16545.05
3*ETW	$\alpha = 1.1047420$ $\lambda = 0.9997629$	8269.085	16546.17	16568.28

Model	Parameter Estimates	−*LL*	AIC	BIC
2*W	$\theta = 1537.5546130$ $\alpha = 0.7969674$ $\theta = 3533.4696971$	8285.724	16575.45	16586.50

The GETW model produced the smallest log-likelihood value (−LL = 8218.032), indicating the best fit to the observed data. The GETW model also yielded the lowest AIC value (16446.06),
which shows the best trade-off between goodness-of-fit and model complexity similarly, the BIC value for the GETW model (16473.70) is the smallest among all competing models, confirming its superiority even under a stronger penalty for model complexity. Based on the log-likelihood, AIC and BIC criteria, the Generalized Exponentiated Transmuted Weibull (GETW) model provides the best representation of the data. The results demonstrate that the additional shape parameter significantly improve the flexibility and modelling capability of the distribution. The GETW model is preferred over the GEW, EW, ETW and classical Weibull models for the analysed survival dataset.

Fitted Density Comparison

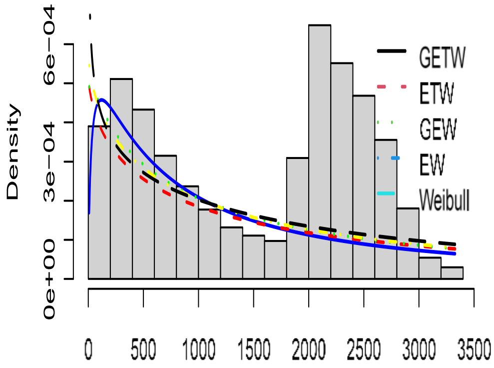

Figure 6: Density Plot

Interpretation of the Fitted Density Comparison Plot

Figure 5 presents the Fitted probability density functions of the competing lifetime models, namely the GETW, ETW, GEW, EW and Weibull distributions. The horizontal axis represents survival time, while the vertical axis shows the estimated probability density. The aim of this plot is to visually assess the ability of each model to capture the underlying distributional pattern of the observed data.

From the Figure, the density curves exhibit a positively skewed shape with a pronounced peak occurring at smaller survival times followed by a gradual decline as time increases. This behavior indicates that a large proportion of failures occur at early times, while relatively few observations survive for long durations. Such a pattern is typical of medical survival and reliability datasets where early failures and long–term survivors coexist.

The GETW density curve closely follows the overall shape of the empirical distribution across the entire range of the data. In particular, the GETW model successfully captures the location of the peak density, the spread of the distribution, and the long right tail behaviour of the data.

The graphical evidence suggests that the GETW distribution provides the best fit among the competing models. The close agreement between the GETW density curve and the empirical distribution demonstrates the advantage of the additional shape parameters in improving model flexibility and capturing complex features of lifetime data.

Discussion

The empirical results consistently showed that the proposed GETW distribution outperformed several existing models, including the Weibull (W), Generalized Weibull (GW), Half-Logistic Generalized Weibull (HLGW), Transmuted Weibull (TW), Exponentiated Weibull (EW), Generalized Exponentiated Weibull (GEW), and Exponentiated Transmuted Weibull (ETW) distributions. The proposed models provided superior goodness-of-fit and greater flexibility in modelling complex survival patterns.

CONCLUSION

This study introduced the GETW distribution, integrating generalization, exponentiation, and transmutation. Defined CDF, PDF, survival, and hazard functions, Derived moments, quantiles, order statistics, Developed MLE procedures and asymptotic properties, Monte Carlo simulation confirmed the Model performance and Real data application showed superior goodness-of-fit compared to existing models GETW is robust and flexible, suitable for engineering, biomedical, and reliability applications. Future work will extend GETW to regression modelling for survival data.

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