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Adomian Decomposition Method for Numerical Solution of 1-D and 2-D Seismic Wave Equations

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ABSTRACT
The concept of real seismic wave equations has several applications through physics, geology, geophysics, and engineering. A precise and effective technique for simulating seismic wave propagation in the Earth's media must be developed to ascertain the Earth's structure. Over the past few decades, wave field simulation has developed into a potent instrument in seismological research, enabling researchers to better understand seismic events and improve predictions of earthquake impacts on various geological structures. The current study employs the well-known Adomian decomposition method (ADM) to solve the one- and two-dimensional seismic wave equations directly to obtain approximate and exact solutions without converting the seismic wave equations into ordinary differential equations (ODEs). The graph of each exact solution of seismic wave equations was plotted to show the movement of various types of seismic wave equations. The obtained results show the applicability and usefulness of the Adomian decomposition method to approximate solutions of seismic wave equations, making it suitable for geophysical applications such as exploration and subsurface imaging.

INTRODUCTION

Since seismology is a data-driven science, its most significant findings typically come from fresh data sets or the creation of novel techniques for data analysis. Seismic waves are basic physical disruptions caused by earthquakes, explosions, or tectonic activity that spread across the interior and crust of the Earth. In order words, the study of earthquakes and seismic waves, which provide information about the structure of the Earth, is known as seismology.

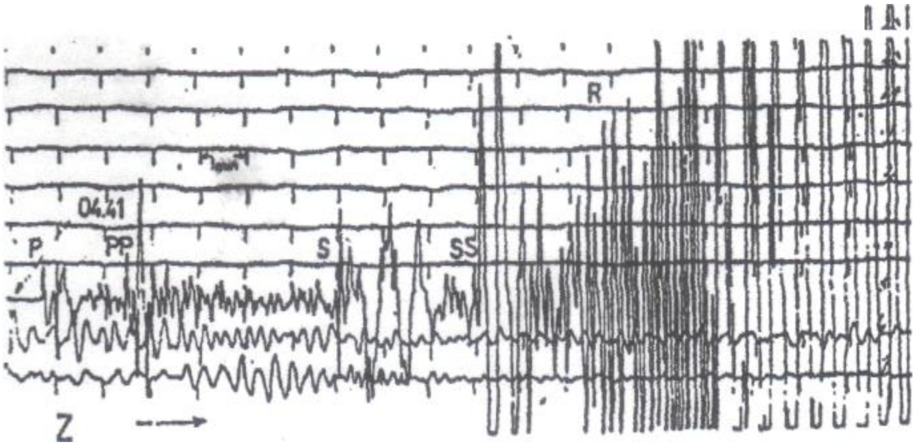

Figure 1: Shows a Typical Example of Seismic Wave Equation Propagation

The mathematical description of seismic wave propagation is generally expressed through wave equations derived from the principles of elasticity and continuum mechanics. These equations model the evolution of displacement, velocity, or stress fields in space and time and form the basis for many problems in geophysics, earthquake engineering, and exploration seismology. A simplified representation of the seismic wave equation can be written as

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2 \nabla^2 u(x,y,t) \quad (1)$$

Where $u(x,t) and\ u(x,y,t)$ represents the displacement field and c is the propagation velocity in the medium. Such equations may also be reformulated into equivalent first-order systems when describing velocity–stress relationships in seismic modeling. The complexity of geological structures and boundary conditions often makes obtaining exact analytical solutions to wave equations extremely difficult. Consequently, many researchers rely on numerical approaches such as finite difference methods, finite element methods, and spectral methods to approximate the solution of wave equations in geophysical models. Although these methods have proven highly useful, they require extensive computational resources and may introduce numerical errors due to discretization or stability constraints.

To address these limitations, semi-analytical techniques have gained considerable attention in applied mathematics. Among these methods, the Adomian Decomposition Method (ADM) has become a widely used approach for solving linear and nonlinear differential equations. ADM was originally introduced by George Adomian as a technique that decomposes the solution of differential equations into a rapidly converging series without requiring discretization or linearization of the governing equations. The solution is expressed as a series

$$\left.\begin{aligned} u(x,t) &= \textstyle\sum_{n=0}^{\infty} u_n(x,t) \\ u(x,y,t) &= \textstyle\sum_{n=0}^{\infty} u_n(x,y,t) \end{aligned}\right\} \quad (2)$$

Where the components $u_n(x,y,t)$ are obtained recursively, while nonlinear terms are represented using specially constructed Adomian polynomials. This decomposition framework enables analytical approximations to be derived with minimal computational effort.

Over the years, ADM has been applied successfully to a wide variety of mathematical models, including nonlinear oscillators, reaction–diffusion systems, fluid dynamics equations, and wave propagation problems. The method has been particularly effective in solving partial differential equations because it avoids perturbation parameters and discretization schemes that are typically required by classical numerical techniques.

In recent years, significant progress has been made in improving and extending the Adomian decomposition method. For example, Alkarawi and Al-Saiq (2020) applied ADM to nonlinear wave equations and the Klein–Gordon equation, demonstrating that the method provides accurate series solutions with fast convergence for nonlinear wave models. Their results confirmed that ADM can effectively capture nonlinear wave dynamics while maintaining analytical structure.

Further developments have focused on applying ADM to multidimensional and nonlinear wave equations. Dehraj (2023) investigated a two-dimensional nonlinear wave equation with derivative and power nonlinearities using ADM and the variational iteration method. The study showed that ADM produces highly accurate approximate solutions, with convergence observed up to several decimal places when compared with exact analytical results. These findings highlight the robustness of ADM in solving nonlinear partial differential equations.

3-D frequency-domain seismic wave modelling in heterogeneous, anisotropic media using a Gaussian quadrature grid approach by Bing Zhou and Greenhalgh (2015). They produced two numerical solutions in full-space homogeneous, isotropic and anisotropic media, respectively, and compare them with the analytical solutions, as well as showed the excellent effectiveness of the PML model parameters. In addition, they performed numerical simulations for 3-D seismic waves in a heterogeneous, anisotropic model incorporating a free-surface ridge topography and validated the results against the 2.5-D modelling solution, and demonstrate the capability of the approach to handle realistic situations.

Similarly, Walters et al., (2020) presented a new analytical model was developed in two-dimensional Cartesian coordinates. Combined with an initial condition of sufficient symmetry, this provided a valuable check for the validity of the numerical method that follows. A particular initial condition is found which allows for a new closed-form solution. A numerical scheme is then presented which combines a spectral (Fourier) representation for displacement components and wave-speed parameters, a fourth-order Runge–Kutta integration method, and an absorbing boundary layer. The resulting large system of differential equations was solved in parallel on suitable enhanced performance desktop hardware in a new software implementation. This provides an alternative approach to forward modelling of waves within isotropic media which is efficient, and tailored to rapid and flexible developments in modelling seismic structure, for example, shallow depth environmental applications. Visual comparisons of the analytic solution and the numerical scheme are presented.

Zhang, Yang and Song (2014) make use of nearly analytic exponential time difference (NETD) method for solving the 2D acoustic and elastic wave equations by transformed the seismic wave equations into semi-discrete ordinary differential equations (ODEs), and then, the converted ODE system is solved by the exponential time difference (ETD) method. From their theoretical analyses and numerical results, the NETD can suppress numerical dispersion effectively by using the displacement and gradient to approximate the high-order spatial derivatives. In addition, because NETD was based on the structure of the Lie group method which preserves the quantitative properties of differential equations.

Another important research direction involves modifying the classical ADM to improve its convergence and computational efficiency on fractional differential equations. For instance, Al-Mazmumy et al. (2024) proposed a modified ADM that incorporates orthogonal polynomials such as Legendre and Chebyshev polynomials to solve fractional differential equations with improved accuracy and stability. Their approach demonstrates that combining ADM with approximation theory can significantly enhance the method's performance in solving complex differential equations.

Similarly, Bekela (2024) introduced a hybrid Yang Transform–Adomian Decomposition Method, which integrates the Yang transform with ADM to solve nonlinear time-fractional partial differential equations. The proposed method was shown to exhibit strong convergence properties and high computational efficiency, making it suitable for solving nonlinear wave-type models.

Recent studies have also explored iterative or accelerated versions of ADM for nonlinear wave problems. For example, research published in the journal *Mathematics* demonstrated that both the classical ADM and the Iterative Adomian Decomposition Method (IADM) can accurately approximate nonlinear wave solutions, including shock wave behavior in nonlinear evolution equations. Error analysis revealed that the ADM-based solutions converge rapidly to benchmark analytical solutions, confirming the reliability of the method for nonlinear wave modeling.

In addition to these developments, ADM has been applied to various geophysical and fluid-dynamics wave models. Recent semi-analytical studies using ADM to analyze shallow water wave equations demonstrated that the method can generate accurate approximations while preserving important physical parameters governing wave motion. These results indicate that ADM has strong potential for modeling wave phenomena in complex physical systems.

Despite these advancements, relatively limited research has focused specifically on applying ADM to seismic wave equations, particularly when considering both first-order and second-order formulations of the governing equations. Most existing studies have examined general wave equations or nonlinear evolution equations rather than seismic models that describe wave propagation in elastic media. Furthermore, comparative studies analyzing the performance of ADM across different orders of seismic wave equations remain scarce.

To improve the computational accuracy and efficiency of wave field simulation, many numerical techniques have been developed. Currently, widely used methods include the finite difference method (FDM) (Blanch and Robertson 1997, Carcione and Helle 1999, Dablain 1986, Igel et al. 1995, Kelly et al. 1976), the finite element method (Erikson and Johnson 1991), the pseudo-spectral method (PSM) (Kosloff and Baysal 1982), the spectral element method (Komatitsch and Vilotte 1998, Komatitsch et al. 2000), the reflectivity method (Booth and Crampin 1983 a, b, Chen 1993), and the boundary integral equation-discrete wavenumber method (Bouchon 1996, Zhou and Chen 2008). Each method has its own advantages and disadvantages, which we have reviewed briefly in our previous work (Yang et al. 2004). This study uses the Adomian decomposition method to investigate the approximate solutions to the 1D and 2D seismic wave equations. Examining an approximate solution to the 1D and 2D seismic wave problem using a semi-analytic approach of interest in the quickly converging Adomian Decomposition Methods (ADM) is the study's specific goal and may offer an efficient alternative to conventional numerical approaches used in geophysical simulations.

MATERIALS AND METHODS

The Description of the Adomain Decomposition Method

The use of the Adomian decomposition method has been applied to a wide class of problems in the sciences. (Wazwaz 2000). The Adomian Decomposition Method is useful for obtaining the closed form and numerical approximations of linear formal solutions to a wide class of stochastic and deterministic problems in science and engineering involving algebraic, differential, and integro-differential equations. This method was introduced by the American mathematician George Adomian (1923-1986) in search of a solution in the form of a series and on decomposing the non-linear operator into a series in which the terms are calculated recursively using Adomian polynomials (Adomian, 1994; Adomian, 1988) and (Adomian and Rach, 1991).

To solve one- and two-dimensional seismic wave equations using the Adomian Decomposition Method, first we rewrite it in the following form (Wazwaz, 2000; Jiao, Dang, & Yamamoto, 2008);

$$Lu(x,y,t) + Nu(x,y,t) + Ru(x,y,t) = g(x,y,t) \tag{3}$$

Where the operators $L, N\ and\ R$ are invertible linear, nonlinear and the remaining linear operators respectively. Also g is a known function. We are looking to obtain the unknown function$u(x,y,t)$. By applying the inverse operator L^{-1} to the both sides of equation (3), we have:

$L^{-1}Lu(x,y,t) + L^{-1}Nu(x,y,t) + L^{-1}Ru(x,y,t) = L^{-1}g(x,y,t)$ (4)

We know that $L^{-1}Lu(x,y,t) = u(x,y,t) + k$, where k is occurred from the integrations. So, it leads:

$u(x,y,t) = L^{-1}g(x,y,t) - L^{-1}Nu(x,y,t) - L^{-1}Ru(x,y,t) - k$ (5)

From the ADM, we decomposed $u(x,y,t)$ and $Nu(x,y,t)$ in the infinite series as

$u_n = \sum_{n=0}^{\infty} u_n, Nu(x,y,t) = \sum_{n=0}^{\infty} A_n$ (6)

Where A_n called Adomian polynomials which could be calculated by Wazwaz (2000)

$A_n(u_0, u_1, u_2, \dots, u_n) = A_n = \frac{1}{n!}\left\{\frac{d^n}{d\lambda^n}\right\} N\{\sum_{i=0}^{\infty} \lambda^i U_i\}_{\lambda \neq 0}$ (7)

To obtain $u_n, n = 0(1)\infty$, we substitute (6) in (5) then we chose $u0$ as $L^{-1}g(x,y,t) - k$ and

Other terms of $u(x,y,t)$ will calculated by the following recurrence formula as

$u_{n+1} = -L^{-1}A_n - L^{-1}Ru_n(x,y,t), \quad n = 0(1)\infty$ (8)

The partial sum of the series is thus obtained as

$U(x,y,t) = \sum_{n=0}^{\infty} U_n(x,y) = \{u_0, u_1, u_2, \dots, u_\infty\}$ (9)

Note that (9) lead to exact solution.

RESULTS AND DISCUSSION

Derivation of One-Dimensional Seismic Wave Equations and their solution

Here, we consider Newton's second law of motion mathematical equation given by

$F = ma = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}$ (10)

For a string element displaced in the y direction then the next vertical force can be written as;

$f_y = \{\sin(\theta_2) - \sin(\theta_1)\}$ (11)

If the angles are small then

$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ (12)

And

$\tan(\theta) = \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots$ (13)

Hence,

$\sin(\theta)\tan(\theta)\frac{\partial u_y}{\partial x} = \left\{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right\}\left\{\frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots\right\}\frac{\partial u_y}{\partial x}$ (14)

$f_y \cong \left\{\left(\frac{\partial u_y}{\partial x}_{x \to \Delta x} - \frac{\partial u_y}{\partial x}_x\right) = \Delta y\right\}$ (15)

Where,

$\Delta y = \frac{\partial u_y}{\partial x}_{x \to \Delta x} - \frac{\partial u_y}{\partial x}_x$ (16)

is the change in slope. Newton's second law of motion can now be written as

$f_y = ma = \rho^{\Delta x}\frac{\partial^2 u_y}{\partial t^2} = \omega \Delta y$ (17)

Hence,

$\rho^{\Delta x}\frac{\partial^2 u_y}{\partial t^2} = \omega \Delta y$ (18)

So that,

$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{\partial^2 u_y}{\partial x^2}$ (19)

Therefore, (10) becomes 1D Seismic Wave Equation written as

$\frac{\partial^2 u_y}{\partial t^2} = \frac{\omega}{\rho}\frac{\partial^2 u_y}{\partial x^2}$ (20)

Where $c = \sqrt{\frac{\omega}{\rho}}$

Where u is the displacement function, the harmonic solution is given by

$U(x, t) = A\sin(kx - \omega t) = A\sin(kt - kct)$

Where $k = \frac{2\pi}{\lambda}$ is the wave number, λ is the wave length and $\omega = 2\pi f$ is the angular frequency, c is the phase velocity or the radian frequency

Numerical Examples

In this section, we solve a few 1-D and 2-D seismic wave equations (SWE) by the Adomian decomposition method. A second-order linear partial differential equation, the wave equation describes a variety of waves, including water, light, and sound waves. The phenomena of a vibrating string, which displays oscillations in one and two dimensions, are the subject of this section. Let $u(x,t)$ and $u(x,y,t)$ represent the displacement of the string from its equilibrium (horizontal) position of 1D and 2D seismic wave equations, where x and y are the position along the string and t is time. The displacement $u(x,t)$ and $u(x,y,t)$ are limited to one dimension and vary with both position and time. The constant c and the function $f(x)$ must be prescribed.

Numerical Example 1: we consider one-dimension of seismic wave equations

$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2} = 0, -\infty < x < \infty, \ t > 0, U(x,0) = sinx, \ u_t(x,t) = 0.$

The operator L is $L = \frac{d}{dt}$, by applying L^{-1}, we obtain,

$u_t(x,t) = u_t(x,0) + \int_0^t u_{xx}(x,t)dt = \int_0^t u_{xx}(x,t)dt$

Similarly, in a view of (4), we have

$u(x,t) = u(x,0) + \iint_0^t u_{xx}(x,t)dtdt = sinx + \iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt$

Where

$u_{n+1}(x,t) = \iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt, \ n = 1(1)\infty$

And

$u_0(x,t) = sinx$

The first limited terms are

$$\left.\begin{aligned} u_1(x,t) &= \iint_0^t \frac{\partial^2 u_0(x,t)}{\partial x^2} dtdt = -\frac{t^2}{2!} sinx \\ u_2(x,t) &= \iint_0^t \frac{\partial^2 u_1(x,t)}{\partial x^2} dtdt = \frac{t^4}{4!} sinx \\ u_3(x,t) &= \iint_0^t \frac{\partial^2 u_1(x,t)}{\partial x^2} dtdt = \frac{t^6}{6!} sinx \\ &\vdots \\ u_{n+1}(x,t) &= \iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2} dtdt = -\frac{t^n}{n!} sinx, n\ is\ an\ even\ number. \end{aligned}\right\} \quad (21)$$

An approximate solution gives

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots u_\infty(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$

By simplifying, we obtain

$$u(x,t) = sinx\left\{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right\} = \sin(x).\cos(t) \quad (22)$$

Conclusively, 1D-seismic wave equation gives exact solution as

$u(x,t) = \sin(x).\cos(t)$

Table 1: shows the Global Absolute Errors (GAE) of 1D Seismic Wave Equation

$\boldsymbol{U(x,t)}$	$\boldsymbol{App\ Sol = U_{(x,t)}}$	$\boldsymbol{Exact\ Sol = U(x,t)}$	GAE $= \left\lvert \boldsymbol{U(x,t) - U_{(x,t)}} \right\rvert$
$\boldsymbol{t = 0.1}$	1.7365210×10^{-2}	1.7452379×10^{-2}	$8.716200000 \times 10^{-5}$
$\boldsymbol{t = 0.2}$	1.7104520×10^{-2}	1.7452300×10^{-2}	$3.477800000 \times 10^{-4}$
$\boldsymbol{x = 1, t = 0.3}$	1.6672920×10^{-2}	1.7452167×10^{-2}	$7.792470000 \times 10^{-4}$
$\boldsymbol{t = 0.4}$	1.6074730×10^{-2}	1.7451981×10^{-2}	$1.377251000 \times 10^{-4}$
$\boldsymbol{t = 0.5}$	1.5315927×10^{-2}	1.7451741×10^{-2}	$2.135813460 \times 10^{-3}$
$\boldsymbol{t = 0.1}$	3.4725144×10^{-2}	3.4899443×10^{-2}	$1.742984200 \times 10^{-4}$
$\boldsymbol{t = 0.2}$	3.4203830×10^{-2}	3.4899284×10^{-2}	$6.956127200 \times 10^{-4}$
$\boldsymbol{x = 2, t = 0.3}$	3.3340762×10^{-2}	3.4899018×10^{-2}	$1.558255352 \times 10^{-3}$
$\boldsymbol{t = 0.4}$	3.4624003×10^{-2}	3.4898646×10^{-2}	$2.746424600 \times 10^{-4}$
$\boldsymbol{t = 0.5}$	3.0627890×10^{-2}	3.4898167×10^{-2}	$4.270977306 \times 10^{-3}$
$\boldsymbol{t = 0.1}$	5.2074494×10^{-2}	5.2335876×10^{-2}	$2.613820000 \times 10^{-4}$
$\boldsymbol{t = 0.2}$	5.1292721×10^{-2}	5.2335637×10^{-2}	$1.042915511 \times 10^{-3}$
$\boldsymbol{x = 3, t = 0.3}$	4.9998448×10^{-2}	5.2335238×10^{-2}	$2.336789314 \times 10^{-3}$
$\boldsymbol{t = 0.4}$	4.8204607×10^{-2}	5.233468×10^{-2}	$4.130072173 \times 10^{-3}$
$\boldsymbol{t = 0.5}$	4.5929122×10^{-2}	5.2333963×10^{-2}	$6.404840488 \times 10^{-3}$

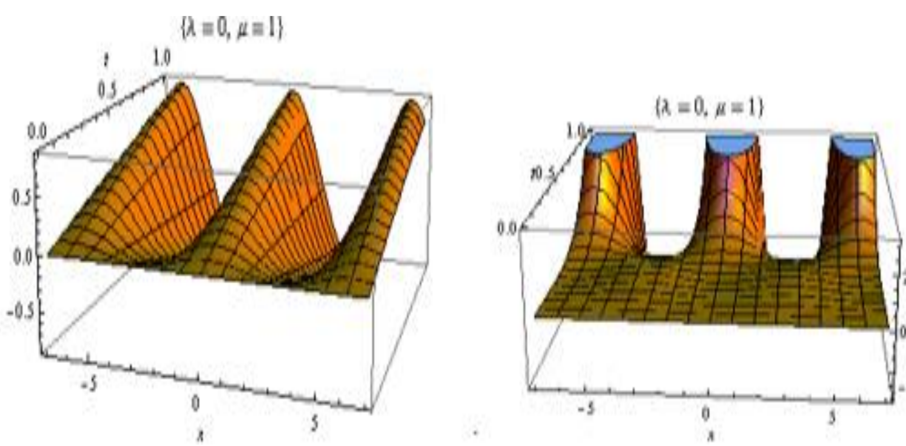

Figure 2: Snapshot of Seismic Wave Displacement with Increasing *X* and *T* for Exact Solution 1

The nature of the graph represents a periodic, oscillatory surface. Likely a wave function $u(x,t) = \sin x \cos t$ give solution to a PDE such as a heat and wave equation. Possibly a traveling wave or evolving sinusoidal profile. In simple terms, it's a smooth, repeating wave surface in 3D, where the shape oscillates along one direction and changes gradually along the other.

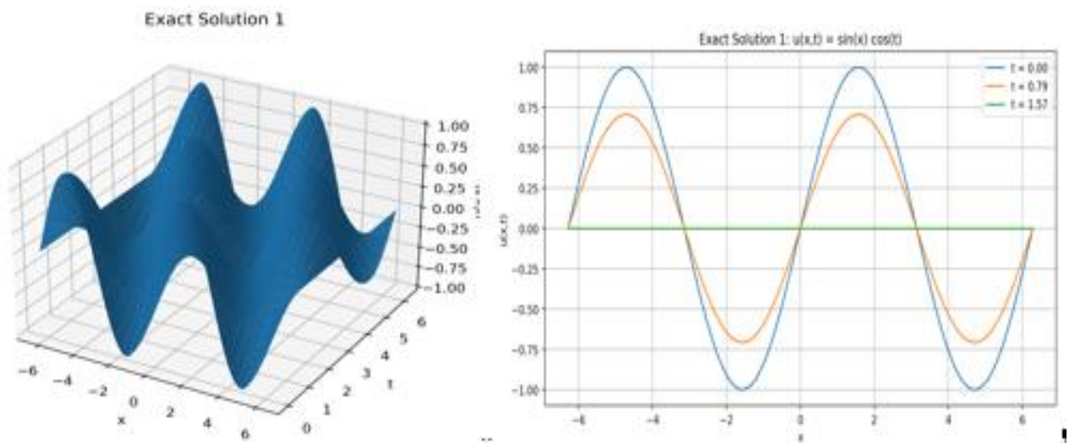

Figure 3: Shows the Surface Plot and Curves' Representation of a Wave-Like Sinusoidal Behaviour in Space and Time of the Exact Solution 1

Generally, the nature of the graph satisfied Sinusoidal in space (x), which means each curve is a sine wave along the x-axis (periodic, smooth oscillations). Time-dependent amplitude, which means the factor cos (t) scales the wave at $t = 0$, give full amplitude. Att ≈ 0.79: amplitude is smaller. At $t \approx 1.57$ ($\approx \pi/2$), cos (t) = 0, the graph becomes a flat line at zero. Standing wave (not traveling), the shape does not shift left or right over time, it only changes height. This is characteristic of a standing wave. Periodic behavior is period in space at 2π and period in time at 2π. In conclusion, this is a classic solution of the wave equation, showing oscillation at fixed positions rather than movement through space.

Numerical Example 2: we consider one-dimension of seismic wave equations with variables coefficient as $\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ -\infty < x < \infty, t > 0,\ U(x,0) = 0,$ $u_t(x,t) = x^2$.

The operator L is $L = \frac{d}{dt}$, by applying L^{-1}, we obtain,

$u_t(x,t) = u_t(x,0) - \int_0^t \frac{x^2}{2} u_{xx}(x,t)dt = x^2 - \int_0^t \frac{x^2}{2} u_{xx}(x,t)dt$

Similarly, in a view of (4), we have

$u(x,t) = u(x,0) + \iint_0^t u_{xx}(x,t)dtdt = x^2 t - \iint_0^t \frac{x^2}{2}\frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt$

Where

$u_{n+1}(x,t) = -\frac{x^2}{2}\iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt,\ n = 1(1)\infty$

And

$u_0(x,t) = x^2 t$

The first few terms are

$$\left.\begin{aligned} u_1(x,t) &= -\frac{x^2}{2}\iint_0^t \frac{\partial^2 u_0(x,t)}{\partial x^2}dtdt = -\frac{t^3}{3!}x^2 \\ u_3(x,t) &= -\frac{x^2}{2}\iint_0^t \frac{\partial^2 u_2(x,t)}{\partial x^2}dtdt = \frac{t^5}{5!}x^2 \\ u_4(x,t) &= -\frac{x^2}{2}\iint_0^t \frac{\partial^2 u_3(x,t)}{\partial x^2}dtdt = -\frac{t^7}{7!}x^2 \\ &\vdots \\ u_{n+1}(x,t) &= \mp\frac{x^2}{2}\iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt = \mp\frac{t^n}{n!}x^2, n\ is\ odd\ number. \end{aligned}\right\} \quad (23)$$

An approximate solution gives

$$u(x,t) = x^2\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right\} = x + x^2\sin(t) \quad (24)$$

Conclusively, 1D-seismic wave equation with variables coefficient gives exact solution;
$u(x,t) = x^2 \sin(t)$.

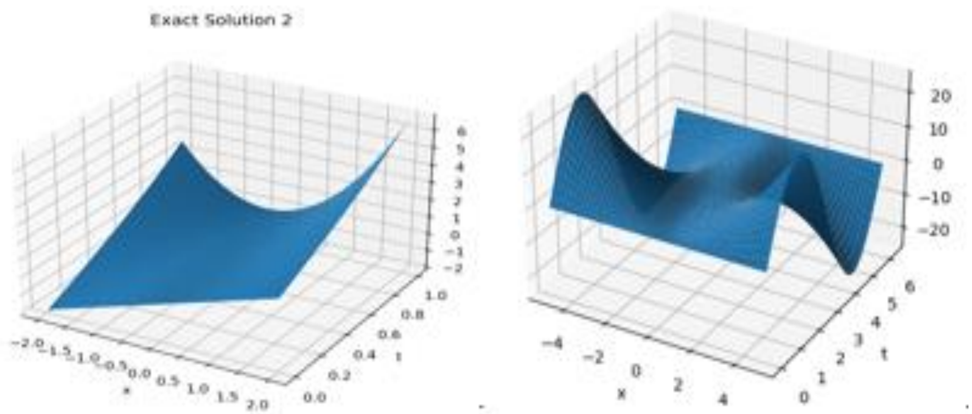

Figure 4: Surface Plot of the Exact Solution 2

This is a 3D surface plot labeled "Exact Solution 2," showing how a quantity varies with both space and time. The nature of the graph shows a type of smooth and continuous curved surface. The behavior shown on the surface looks parabolic and quadratic in that direction. It dips down and then rises, suggesting a minimum point along it. Behavior shows that as increases, the surface generally rises upward. This indicates the solution is growing over time. From overall shape, it resembles a tilted parabolic sheet or a nonlinear evolving surface and likely represents a solution to a PDE such as a heat or diffusion-type equation with both spatial curvature and time growth. Generally, the interpretation shows that the solution is nonlinear and time-dependent; it appears stable and smooth (no oscillations or discontinuities), and the increasing height with time suggests amplification or accumulation rather than decay.

This 3D surface plot, labeled "Exact Solution 2," illustrates how a quantity varies with space and time, forming a smooth, continuous curved surface resembling a tilted parabolic sheet. Its behavior is characterized by a dip followed by a rise, indicating a minimum point, and suggests a solution to a PDE like a heat or diffusion equation. The surface's growth over time reflects stability with nonlinear, time-dependent characteristics, and the increasing height indicates amplification or accumulation rather than decay.

Numerical Example 3: we consider two-dimension of seismic wave equations with variables coefficient as $\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{x^2}{12}\frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{y^2}{12}\frac{\partial^2 u(x,y,t)}{\partial y^2} = 0,\ -\infty < x < \infty, t > 0,\ U(x,0) = x^4,\ \ u_t(x,0) = y^4.$

The operator L is $L = \frac{d}{dt}$, by applying L^{-1}, we obtain,

$$u_t(x,y,t) = u_t(x,y,0) + \int_0^t \frac{x^2}{12} u_{xx}(x,y,t)dt + \int_0^t \frac{y^2}{12} u_{yy}(x,y,t)dt$$

$$\therefore \quad \frac{\partial u(x,y,t)}{\partial t} = y^4 + \int_0^t \left\{\frac{x^2}{12}u_{xx}(x,t) + \frac{y^2}{12}u_{yy}(x,y)\right\} dt$$

Similarly, in a view of (4), we have

$$u(x,y,t) = u(x,y,0) + u_t(x,y,0) + \iint_0^t \left\{\frac{x^2}{12}u_{xx}(x,t) + \frac{y^2}{12}u_{yy}(x,y)\right\} dtdt$$

Then,

$$u(x,y,t) = x^4 + y^4 t + \iint_0^t \left\{\frac{x^2}{12}u_{xx}(x,y,t) + \frac{y^2}{12}u_{yy}(x,y,t)\right\} dtdt,\ n = 1(1)\infty$$

Where

$$u_1(x,y,t) = x^4 + y^4 t,\ u_{n+1}(x,y,t) = \iint_0^t \left\{\frac{x^2}{12}\frac{\partial^2 u_n}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_n}{\partial y^2}\right\} dtdt$$

The first few terms are

$$\left.\begin{aligned} u_2(x,y,t) &= \iint_0^t \left\{\frac{x^2}{12}\frac{\partial^2 u_1}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_1}{\partial y^2}\right\} dtdt = \frac{t^2}{2!}x^4 + \frac{t^3}{3!}y^4 \\ u_3(x,y,t) &= \iint_0^t \left\{\frac{x^2}{12}\frac{\partial^2 u_2}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_2}{\partial y^2}\right\} dtdt = \frac{t^4}{4!}x^4 + \frac{t^5}{5!}y^4 \\ u_4(x,y,t) &= \iint_0^t \left\{\frac{x^2}{12}\frac{\partial^2 u_3}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_3}{\partial y^2}\right\} dtdt = \frac{t^6}{6!}x^4 + \frac{t^7}{7!}y^4 \\ &\vdots \\ u_{n+1}(x,y,t) &= \iint_0^t \left\{\frac{x^2}{12}\frac{\partial^2 u_n}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_n}{\partial y^2}\right\} dtdt = \frac{t^k}{k!}x^4 + \frac{t^q}{q!}y^4, \end{aligned}\right\} \quad (25)$$

Where k is an even number and q is an odd number. An approximate solution gives

$$u(x,t) = y^4\left\{t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right\} + x^4\left\{1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots\right\} \quad (26)$$

Conclusively, 2D-seismic wave equation with variables coefficient gives exact solution;

$$u(x,y,t) = x^4\sinh(t) + y^4\cosh(t)$$

Since the third solution depends on three independent variablesx, y *and* t, it is presented as surface plots over x and y at fixed time t values below.

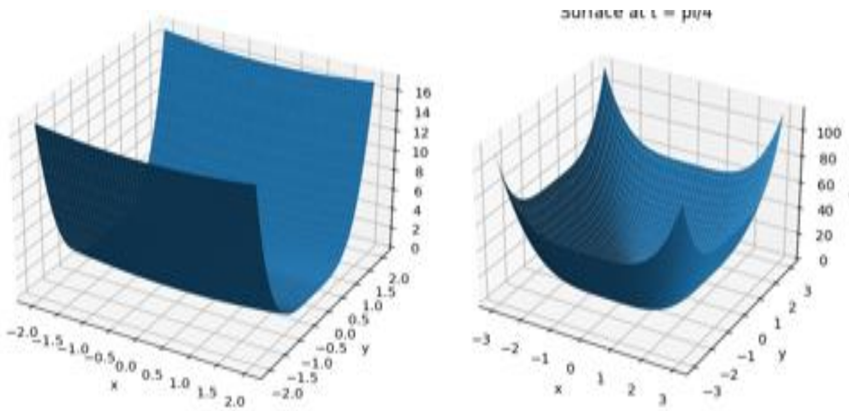

Figure 5: Shows a Snapshot of Wave Displacement with Increasing x and y for Exact Solution 3

The graphs discuss the features of the elliptic paraboloid, emphasizing its single global minimum and symmetry around the vertical axis, as well as noting that the reference to "t = π/4" indicates a specific slice of a parameterized surface while maintaining the overall paraboloid shape.

Numerical Example 4

To further affirm the numerical method of the ADM approach, the following initial 1-D problem was considered (Zhang, *et al*, 2014).

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial x^2}, -\infty < x < \infty, t > 0, U(x,0) = \cos\left\{-\frac{2\pi f_0}{c_0}x\right\}, c_0 = 4000m/s$$

$$u_t(x,t) = -2\pi f_0 sin\left\{-\frac{2\pi f_0}{c_0}x\right\}, \; T = 0.5sec, \; f_0 = 4Hz.$$

The operator L is $L = \frac{d}{dt}$, by applying L^{-1}, we obtain,

$$u_t(x,t) = u_t(x,0) + \int_0^t u_{xx}(x,t)dt = -2\pi f_0 sin\left\{-\frac{2\pi f_0}{c_0}x\right\}. + \int_0^t u_{xx}(x,t)dt$$

Similarly, in a view of (4), we have

$$u(x,t) = u(x,0) + tu_t(x,0) + \iint_0^t u_{xx}(x,t)dtdt$$

$$u(x,t) = \cos\left\{-\frac{2\pi f_0}{c_0}x\right\} - 2t\pi f_0 sin\left\{-\frac{2\pi f_0}{c_0}x\right\} + \iint_0^t u_{xx}(x,t)dtdt$$

Where

$$u_0(x,t) = \cos\left\{-\frac{2\pi f_0}{c_0}x\right\} - 2t\pi f_0 sin\left\{-\frac{2\pi f_0}{c_0}x\right\}, \; u_{n+1}(x,t) = \iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt$$

The first limited terms are

$$\left.\begin{aligned}
u_1(x,t) &= c_0^2\iint_0^t \frac{\partial^2 u_0(x,t)}{\partial x^2}dtdt = c_0^2\left[-\left\{\frac{2\pi f_0}{c_0}\right\}^2\frac{t^2}{2!}cos\left\{-\frac{2\pi f_0}{c_0}\right\}x + \left\{\frac{2\pi f_0}{c_0}\right\}^3\frac{t^3}{3!}sin\left\{-\frac{2\pi f_0}{c_0}\right\}x\right] \\
u_2(x,t) &= c_0^2\iint_0^t \frac{\partial^2 u_1(x,t)}{\partial x^2}dtdt = c_0^4\left[\left\{\frac{2\pi f_0}{c_0}\right\}^4\frac{t^4}{4!}cos\left\{-\frac{2\pi f_0}{c_0}\right\}x + \left\{\frac{2\pi f_0}{c_0}\right\}^4\frac{t^5}{5!}sin\left\{-\frac{2\pi f_0}{c_0}\right\}x\right] \\
u_3(x,t) &= c_0^2\iint_0^t \frac{\partial^2 u_2(x,t)}{\partial x^2}dtdt = c_0^6\left[-\left\{\frac{2\pi f_0}{c_0}\right\}^6\frac{t^6}{6!}cos\left\{-\frac{2\pi f_0}{c_0}\right\}x - \left\{\frac{2\pi f_0}{c_0}\right\}^6\frac{t^7}{7!}sin\left\{-\frac{2\pi f_0}{c_0}\right\}x\right] \\
&\vdots \\
u_{n+1}(x,t) &= c_0^2\iint_0^t \frac{\partial^2 u_n(x,t)}{\partial x^2}dtdt = c_0^r\left[\mp\left\{\frac{2\pi f_0}{c_0}\right\}^p\frac{t^m}{m!}cos\left\{-\frac{2\pi f_0}{c_0}\right\}x \mp \left\{\frac{2\pi f_0}{c_0}\right\}^n\frac{t^q}{q!}sin\left\{-\frac{2\pi f_0}{c_0}\right\}x\right]
\end{aligned}\right\} \quad (21)$$

An approximate solution gives

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots u_\infty(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

By simplifying, we obtain

$$u(x,t) = cos\left\{-\frac{2\pi f_0}{c_0}\right\}x.\left\{1 - (2\pi f_0)^2\frac{t^2}{2!} + (2\pi f_0)^4\frac{t^4}{4!} - \cdots\right\} + sin\left\{-\frac{2\pi f_0}{c_0}\right\}x.\left\{(2\pi f_0)t + (2\pi f_0)^2\frac{t^3}{3!} + (2\pi f_0)^5\frac{t^5}{5!} + \cdots\right\}$$

Conclusively, 1D-seismic wave equation gives exact solution as

$$u(x,t) = \sin\left\{2\pi f_0\left(t - \frac{x}{c_0}\right)\right\}$$

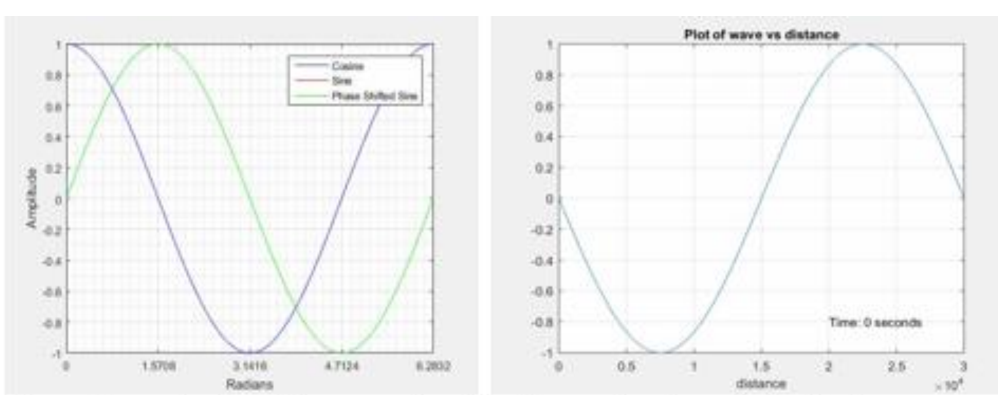

Figure 6: Shows the Visual Interpretation of the Wave and Perfect Relationship between Wave and Distance

Remarks the exact solution $u(x,t) = \cos\left\{2\pi\left(t - \frac{x}{4000}\right)\right\}$ represents a traveling cosine wave moving in the positive x-direction. The term $\left(t - \frac{x}{4000}\right)$ shows that the wave profile shifts over time, meaning the shape of the cosine curve remains the same but translates horizontally as time increases. The coefficient 2π indicates that the wave has a period of 1 unit in time, since cosine completes one full cycle when its argument increases by2π. The factor of $\frac{1}{4000}$ inside the argument determines the wave speed, which is 4000 units per second, implying the wave travels very fast along the spatial axis. At any fixed time t, the graph with respect to x is a cosine curve, and at any fixed position x, the displacement oscillates periodically in time. Overall, this solution is typical of wave equations in physics, especially in describing vibrations, sound waves, or seismic waves propagating through a medium.

Numerical Example 5

To further sustain the numerical method of the ADM approach, the following initial 2-D problem was considered (Zhang, *et al*, 2014).

$$\frac{1}{c^2}\frac{\partial^2 u(x,y,t)}{\partial t^2} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}, -\infty < x < \infty, t > 0,,$$

With initial value condition and others parameter $U(x,y,0) = \cos\left\{-\frac{2\pi f_0}{c_0}(y sin\alpha_0 + x cos\alpha_0)\right\}$, $T = 0.5 sec, f_0 = 4Hz, c_0 = 4000 m/s$.

$$u_t(x,y,0) = -2\pi f_0 sin\left\{-\frac{2\pi f_0}{c_0}(y sin\alpha_0 + x cos\alpha_0)\right\}$$

The operator L is$L = \frac{d}{dt}$, by applyingL^{-1}, we obtain,

$$u_t(x,y,t) = u_t(x,y,0) + \iint_0^t \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) dtdt$$

Similarly, in a view of (4), we have

$$u(x,y,t) = u(x,y,0) + t u_t(x,y,0) + \iint_0^t \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) dtdt$$

$$u(x,y,t) = \cos\{-k(y sin\alpha_0 + x cos\alpha_0)\} - 2t\pi f_0 sin\{-k(y sin\alpha_0 + x cos\alpha_0)\} + \iint_0^t \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) dtdt$$

Where

$$u_0(x,y,t) = \cos\{-k(y sin\alpha_0 + x cos\alpha_0)\} - 2t\pi f_0 sin\{-k(y sin\alpha_0 + x cos\alpha_0)\},$$

$$u_{n+1}(x,y,t) = \iint_0^t \left(\frac{\partial^2 u_n(x,y,t)}{\partial x^2} + \frac{\partial^2 u_n(x,y,t)}{\partial y^2}\right) dtdt$$

The first limited terms are

$$\left.\begin{aligned}
u_1(x,y,t) &= c_0^2 \iint_0^t \left(\frac{\partial^2 u_0(x,y,t)}{\partial x^2} + \frac{\partial^2 u_0(x,y,t)}{\partial y^2}\right) dtdt = -k^2 \frac{t^2}{2!} cos\{-k(x cos\alpha_0 + y sin\alpha_0)\} \\
u_2(x,y,t) &= c_0^2 \iint_0^t \left(\frac{\partial^2 u_1(x,y,t)}{\partial x^2} + \frac{\partial^2 u_1(x,y,t)}{\partial y^2}\right) dtdt = k^4 \frac{t^4}{4!} cos\{-k(x cos\alpha_0 + y sin\alpha_0)\} \\
u_3(x,y,t) &= c_0^2 \iint_0^t \left(\frac{\partial^2 u_2(x,y,t)}{\partial x^2} + \frac{\partial^2 u_2(x,y,t)}{\partial y^2}\right) dtdt = -k^6 \frac{t^6}{6!} cos\{-k(x cos\alpha_0 + y sin\alpha_0)\} \\
&\vdots \\
u_{n+1}(x,y,t) &= c_0^2 \iint_0^t \left(\frac{\partial^2 u_n(x,y,t)}{\partial x^2} + \frac{\partial^2 u_n(x,y,t)}{\partial y^2}\right) dtdt = \mp k^m \frac{t^m}{m!} cos\{-k(x cos\alpha_0 + y sin\alpha_0)\}
\end{aligned}\right\}$$

$$where\ k = \frac{2\pi f_0}{c_0}$$

An approximate solution gives

$$u(x,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots u_\infty(x,y,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

By substituting, we obtain

$$u(x,y,t) = \cos\left\{-\left\{\frac{2\pi f_0}{c_0}\right\}(y sin\alpha_0 + x cos\alpha_0)\right\}.\left\{1 + \left\{\frac{2\pi f_0}{c_0}\right\}^2 \frac{t^2}{2!} - \left\{\frac{2\pi f_0}{c_0}\right\}^4 \frac{t^4}{4!} + \left\{\frac{2\pi f_0}{c_0}\right\}^6 \frac{t^6}{6!} - \cdots\right\} - 2t\pi f_0 sin\left\{-\left\{\frac{2\pi f_0}{c_0}\right\}(y sin\alpha_0 + x cos\alpha_0)\right\}.$$

Conclusively, 2-D seismic wave equation gives exact solution as

$$u(x,y,t) = \sin\left\{2\pi f_0\left(t - \frac{x}{c_0} cos\alpha_0 - \frac{y}{c_0} sin\alpha_0\right)\right\}.$$

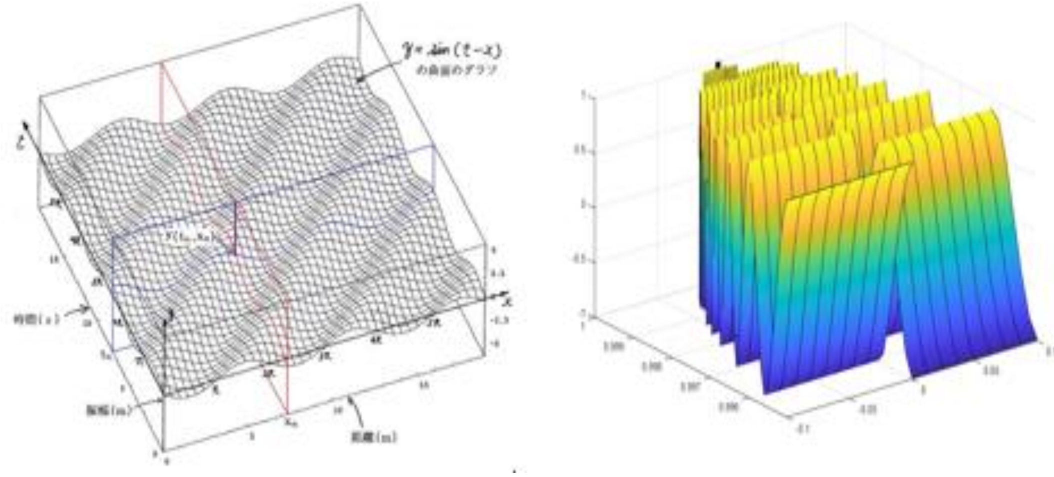

Figure 7: Shows Visual Interpretation of the Exact Solution 5

The nature of the surface in the figure shows rapid oscillations (wave-like ridges) along one direction, while the other direction changes only slightly. This suggests the exact solution is highly oscillatory, like sine and cosine behavior. The oscillations may be compressed or modulated, which means possibly indicating something like a high-frequency wave or a function with a steep gradient in one variable. The "stacked ridge" appearance often comes from functions with sharp periodic transitions or phase wrapping. Conclusively, this kind of graph commonly appears in signal physics processing (e.g., high-frequency waveforms), physics (wave interference or phase functions), and mathematics (trigonometric or Fourier-based functions).

The exact solution $u(x, y, t) = \cos\left\{15\pi \sin\left(t - \frac{x}{4000}\cos\left(\frac{\pi}{4}\right) - \frac{y}{4000}\sin\left(\frac{\pi}{4}\right)\right)\right\}$ d describes a two-dimensional traveling wave with nonlinear modulation. The inner expression $\left\{t - \frac{x}{4000}\cos\left(\frac{\pi}{4}\right) - \frac{y}{4000}\sin\left(\frac{\pi}{4}\right)\right\}$ represents a wave propagating diagonally in the plane $x - y$ at an angle of $\left(\frac{\pi}{4}\right)$, meaning the motion is equally distributed along both spatial directions. The factors $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$, each equal to $\frac{\sqrt{2}}{2}$, determine how the wave speed is resolved along the x and y axes, while the denominator, 4000, controls the overall propagation speed. Unlike a simple cosine wave, the presence of $\sin\left(\frac{\pi}{4}\right)$ something inside the cosine creates a nested or modulated oscillation, causing the wave amplitude pattern to vary in a more complex, almost rippling manner rather than a smooth sinusoid. The coefficient 15π increases the frequency of oscillation inside the cosine, producing rapid fluctuations in the wave profile. Overall, this solution represents a directional, high-frequency, modulated wave, often seen in advanced wave mechanics such as seismic wave interactions or interference patterns in physics.

CONCLUSION

In conclusion, seismic waves help us understand what's happening deep inside the Earth. Since we cannot go down there ourselves, these waves act like messengers, giving us clues about what lies beneath the surface. As scientists, we study how they move and change, so that we can map underground structures. This helps us stay safer during earthquakes, find important resources like oil and minerals and build stronger buildings. Without seismic waves, it would be much harder for us to understand the Earth or predict ourselves from natural disasters. This study used the Adomian decomposition method to solve a set of seismic wave equations in one and two dimensions. The outcomes show the method's convergence and dependability by offering both exact and approximate solutions without additional methods. The $1 - D$ seismic wave equation's numerical and absolute solutions produce a precise result that shows the Adomian Decomposition Method's (ADM) dependability and convergence. The technique uses a graphical representation of the exact solutions $1 - D$ and $2 - D$ to illustrate different types of seismic wave movement, specifically focusing on wave propagation that is periodic and oscillatory in nature at the surface.

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