THE PARALLELISM BETWEEN THE HOWOSU RICCI SCALAR AND THE SCHWARZSCHILD RICCI SCALAR.

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ABSTRACT
In this paper, new Ricci scalar using the Howusu metric tensor was derived which is valid for a gravitational field that is regular and continuous everywhere including all boundaries, continuous normal derivative everywhere including all boundaries, and its reciprocal decreases at infinite distance from the source. The results were compared with the well-known Schwarzschild Ricci Scalar using radial distance as the measuring index and it was discovered that, unlike the Schwarzschild Ricci Scalar, the Howusu metric tensor has a non-zero Ricci scalar as r tends to zero. Comparing also the Howusu Ricci Scalar R with the Schwarzschild Ricci Scalar R, it was found in this paper that, as $r \to 0$, $R = -\infty$ and as $r \to \infty$, R tends to zero, and for the Schwarzschild Metric Tensor, the Ricci Scalar R is $R = 0$. This Ricci scalar is a scalar quantity used to identify and make concrete the notion of curvature for all gravitational fields in nature.
Keywords: Ricci Scalar, Howusu Metric Tensor, Schwarzschild Metric Tensor.

INTRODUCTION

The Schwarzschild metric represents the gravitational field around a symmetrically spherical object without angular momentum. This Schwarzschild metric can be summarized as (Kumar, 2009)

$$g_{\mu v} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{2M}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \quad (1)$$

where M is the mass of the object and r is the distance away from the object. In 2012, Howusu Metric Tensor was introduced and was said to be a solution to Einstein's field equations that describes the gravitational field for all gravitational field in nature. The metric is written as (Howusu, 2012)

$$g_{\mu v} = \begin{pmatrix} -exp\left(\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & exp\left(-\frac{2GM}{c^2 r}\right) & 0 & 0 \\ 0 & 0 & r^2 exp\left(-\frac{2GM}{c^2 r}\right) & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta exp\left(-\frac{2GM}{c^2 r}\right) \end{pmatrix} \quad (2)$$

where C is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object. In this paper, the Ricci Scalar in the Spherical coordinate based upon the Howusu Metric Tensor will be derived, and the results will be compared with the well-known Ricci Scalar in the Spherical Coordinate based on Schwarzschild Metric Tensor.

THEORY

The process of computing the Ricci Scalar involves finding the unknown components of the equation (Kumar, 2009);

$$R = g^{\mu\nu}R_{\mu\nu} \qquad (3)$$

$g_{\mu\nu}$ is the metric tensor to work with, which in this case is the Howusu Metric Tensor. So, the first step is to find the Ricci curvature tensor $R_{\mu\nu}$. This involves finding the Christoffel symbols Γ or the affine connection which only requires one to take the gradient of the entire metric, and partial derivatives of each coefficient of the metric with respect to the components of the metric. Thus, using (Obaje & Ekpe, 2020):

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right), \qquad (4)$$

There are a total of 64 Christoffel symbols Γ since λ, μ and ν each represent four components of the matrix: t, r, θ and $\emptyset$. So, the full set of the Christoffel Symbols must contain a combination of all the possibilities of these coordinates. However, one interesting property of Christoffel Symbols is (Kumar, 2009):

$$\Gamma^{\delta}_{\beta\gamma} = \Gamma^{\delta}_{\gamma\beta} \qquad (5)$$

Setting, $\alpha = 0$, $\beta = 1$ and $\gamma = 0$, then substituting these values into equation (4), we get

$$g_{0\delta}\Gamma^{\delta}_{10} = \frac{1}{2}\left(\frac{\partial g_{01}}{\partial x^{0}} + \frac{\partial g_{00}}{\partial x^{1}} - \frac{\partial g_{10}}{\partial x^{1}}\right) \qquad (6)$$

Substituting equation (2) into equation (6), we arrive at

$$g_{0\delta}\Gamma^{\delta}_{10} = \frac{1}{2}\left(\frac{\partial(0)}{\partial t} + \frac{\partial\left(-exp\left(\frac{2GM}{c^2 r}\right)\right)}{\partial r} - \frac{\partial(0)}{\partial r}\right) \qquad (7)$$

Differentiating equation (7), we obtained

$$g_{0\delta}\Gamma^{\delta}_{10} = \frac{1}{2}\left(-\frac{2GM}{c^2 r^2}exp\left(\frac{2GM}{c^2 r}\right)\right) \qquad (8)$$

$$g_{0\delta}\Gamma^{\delta}_{10} = -\frac{GM}{c^2 r^2}exp\left(\frac{2GM}{c^2 r}\right) \qquad (9)$$

Summing over all the values of δ

$$g_{0\delta}\Gamma^{\delta}_{10} = g_{00}\Gamma^{0}_{10} + g_{01}\Gamma^{1}_{10} + g_{02}\Gamma^{2}_{10} + g_{03}\Gamma^{3}_{10} \qquad (10)$$

Referring to equation (2), the values g_{01}, g_{02} and g_{03} are all zero except for g_{00}

$$g_{0\delta}\Gamma^{\delta}_{10} = g_{00}\Gamma^{0}_{10} + 0 + 0 + 0 \qquad (11)$$

$$g_{00}\Gamma^{0}_{10} = -\frac{GM}{c^2 r^2}\ exp\left(\frac{2GM}{c^2 r}\right) \qquad (12)$$

Multiply both sides by

$$g^{00} = -exp\left(-\frac{2GM}{c^2 r}\right)\Gamma^{0}_{10} = \Gamma^{0}_{01} = \frac{GM}{c^2 r^2} \qquad (13)$$

Similarly, the mathematical results for the other non-zero terms from the calculation of the Christoffel symbols using the Howusu Metric Tensor are:

$$\Gamma^{0}_{10} = \Gamma^{0}_{01} = \frac{GM}{c^2 r^2} \qquad (14)$$

$$\Gamma_{00}{}^{1} = -\frac{GM}{c^2 r^2} \qquad (15)$$

$$\Gamma^{1}_{11} = -\frac{GM}{c^2 r^2} \qquad (16)$$

$\Gamma^1_{22} = \frac{GM}{c^2} - r$ (17)

$\Gamma^1_{33} = \frac{GM}{c^2} sin^2\theta - rsin^2\theta$ (18)

$\Gamma^2_{21} = \frac{1}{\mathrm{r}} - \frac{GM}{c^2\mathrm{r}^2}$ (19)

$\Gamma^2_{33} = -\cos\theta\sin\theta$ (20)

$\Gamma^3_{13} = \frac{1}{r} - \frac{GM}{c^2\mathrm{r}^2}$ (21)

$\Gamma^3_{23} = \frac{\cos\theta}{\sin\theta} = \cot\theta$ (22)

With the Christoffel symbols found, one moves on to find the Riemann curvature tensor $R^{\alpha}_{\mu\alpha\nu}$, using the formula (Obaje & Ekpe, 2021).,

$R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$ (23)

Substituting equations (14) to (22) into equation (23), differentiating where necessary and writing out all the non-zero terms of the Riemann curvature tensor,

Let's look at R^0_{101}.

Making α = o, μ = 1 and ν = 1,then substituting into equation (23), we obtained

$R^0_{101} = \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^0_{\beta 0}\Gamma^{\beta}_{11} - \Gamma^0_{\beta 0}\Gamma^{\beta}_{10}$ (24)

Summing over all the values of β, we have

$R^0_{101} = \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^0_{00}\Gamma^0_{11} + \Gamma^0_{10}\Gamma^1_{11} + \Gamma^0_{20}\Gamma^2_{11} + \Gamma^0_{30}\Gamma^3_{11} - \Gamma^0_{01}\Gamma^0_{10}$
$-\Gamma^0_{11}\Gamma^1_{10} - \Gamma^0_{21}\Gamma^2_{10} - \Gamma^0_{31}\Gamma^3_{10}$ (25)

Substituting the Christoffel symbols in equation (25) and differentiating with respect to the coordinate where necessary, we arrived at

$R^0_{101} = 0 - \frac{2GM}{c^2r^3} + 0 + \left[\left(-\frac{2GM}{c^2r^3}\right)\left(-\frac{2GM}{c^2r^2}\right)\right] + 0 + 0 - \left[\left(-\frac{2GM}{c^2r^2}\right)\left(-\frac{2GM}{c^2r^2}\right)\right] - 0 - 0 - 0$ (26)

$R^0_{101} = -\frac{2GM}{c^2r^3} + \frac{2GM}{c^4r^4} - \frac{2GM}{c^4r^4}$ (27)

$R^0_{101} = -\frac{2GM}{c^2r^3}$ (28)

Similarly, we can obtain the mathematical results for the other non-zero terms from the calculation of Riemann Tensor and it should be noted that only half the amount of Riemann Tensor will be listed because of the following property (Obaje & Ekpe,2021).:

$R^{\alpha}_{\mu\alpha\nu} = -R^{\alpha}_{\mu\nu\alpha}$ (29)

As a result,

$R^0_{202} = \frac{GM}{c^2r} - \frac{G^2M^2}{c^4r^2}$ (30)

$R^0_{303} = \frac{GM}{c^2r} sin^2\theta - \frac{G^2M^2}{c^4r^3} sin^2\theta$ (31)

$R^1_{010} = \frac{2GM}{c^2r^3}$ (32)

$R^1_{212} = 1 - \frac{GM}{c^2r}$ (33)

$R^1_{313} = -\frac{GMsin^2\theta}{c^2r}$ (34)

$R^2_{020} = \frac{GM}{c^4r^3} - \frac{G^2M^2}{c^4r^4}$ (35)

$R^2_{121} = -\frac{GM}{c^2r^3}$ (36)

$R^2_{323} = \frac{2GMsin^2\theta}{c^2r} - \frac{G^2M^2sin^2\theta}{c^4r^2} - cos^2\theta$ (37)

$R^3_{030} = \frac{GM}{c^4r^3} - \frac{G^2M^2}{c^4r^4}$ (38)

$R^3_{131} = -\frac{GM}{c^2r^3}$ (39)

$R^3_{232} = -\frac{G^2M^2}{c^4r^2}$ (40)

Having derived the Riemann curvature tensor, one can now derive the Ricci tensor, which is a contraction of the Riemann tensor (Abalaka & Ekpe, 2021).:

$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ (41)

where $R_{\mu\nu}$ is expressed explicitly as

$R_{\mu\nu} = R^0_{\mu0\nu} + R^1_{\mu1\nu} + R^2_{\mu2\nu} + R^3_{\mu3\nu}$ (42)

Substituting equations (30) to (40) into equation (42) it was discovered that the Howusu Metric Tensor has non-zero Ricci curvature tensor unlike the Schwarzschild Metric Tensor. Thus, the components of the Ricci curvature tensor can be conveniently expressed as follows: (Abalaka & Ekpe,2021)

$R_{00} = \frac{2GM}{c^2r^3} - \frac{2G^2M^2}{c^4r^4} + \frac{2GM}{c^4r^3}$ (43)

$R_{11} = -\frac{4GM}{c^2r^3}$ (44)

$R_{22} = 1 - \frac{2G^2M^2}{c^4r^2}$ (45)

$R_{33} = \frac{2GMsin^2\theta}{c^2r} - \frac{G^2M^2sin^2\theta}{c^4r^3} - \frac{G^2M^2sin^2\theta}{c^4r^2} - cos^2\theta$ (46)

Following this step, the Ricci scalar, R, is calculated by using the formula (Abalaka & Ekpe, 2021).:

$R = g^{\mu\nu}R_{\mu\nu}$ (47)

where $g^{\mu\nu}$ is the contravariant metric tensor; the inverse of the Howusu Metric Tensor is given by (Howusu, 2012):

$g^{00} = -exp\left(\frac{2GM}{c^2r}\right)$ (48)

$g^{11} = exp\left(-\frac{2GM}{c^2r}\right)$ (49)

$g^{22} = \frac{1}{r^2}exp\left(-\frac{2GM}{c^2r}\right)$ (50)

$g^{33} = \frac{1}{r^2sin^2\theta}exp\left(-\frac{2GM}{c^2r}\right)$ (51)

$g^{\mu\nu} = 0$, otherwise (52)

Using the summation convention with the Ricci Scalar, one obtains the value of each coefficient in the expansion: (Abalaka & Ekpe,2021)

$R = g^{00}R_{00} + \cdots + g^{11}R_{11} + \cdots + g^{22}R_{22} + \cdots + g^{33}R_{33}$ (53)

Hence substituting equations (48) to (52) and equations (43) to (46) into equation (53),

$$R = \exp\left(\frac{2GM}{c^2r}\right)\left[\frac{2G^2M^2}{c^4r^4} - \frac{2GM}{c^2r^3} - \frac{2GM}{c^4r^3}\right] + \exp\left(-\frac{2GM}{c^2r}\right)\left[-\frac{2GM}{c^2r^3} + \frac{1}{r^2} - \frac{3G^2M^2}{c^4r^4} - \frac{G^2M^2}{c^4r^5} - \frac{cot^2\theta}{r^2}\right] \quad (54)$$

RESULTS AND DISCUSSION

The mathematical result from the calculation of the Ricci scalar using the Howusu Metric Tensor was:

$$R = \exp\left(\frac{2GM}{c^2r}\right)\left[\frac{2G^2M^2}{c^4r^4} - \frac{2GM}{c^2r^3} - \frac{2GM}{c^4r^3}\right] + \exp\left(-\frac{2GM}{c^2r}\right)\left[-\frac{2GM}{c^2r^3} + \frac{1}{r^2} - \frac{3G^2M^2}{c^4r^4} - \frac{G^2M^2}{c^4r^5} - \frac{\cot^2\theta}{r^2}\right] \qquad (55)$$

where c is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object. Comparing the Ricci Scalar R in (55) with the Schwarzschild Ricci Scalar R, it was found in this paper that, as $r \to 0$, $R = -\infty$ and as $r \to \infty$, R tends to zero, and for the Schwarzschild Metric Tensor, the Ricci Scalar R is

$$R = 0 \qquad (56)$$

Equating the Ricci Scalar of the Howusu Metric Tensor and the Schwarzschild Metric Tensor yields:

$$\exp\left(\frac{2GM}{c^2r}\right)\left[\frac{2G^2M^2}{c^4r^4} - \frac{2GM}{c^2r^3} - \frac{2GM}{c^4r^3}\right] + \exp\left(-\frac{2GM}{c^2r}\right)\left[-\frac{2GM}{c^2r^3} + \frac{1}{r^2} - \frac{3G^2M^2}{c^4r^4} - \frac{G^2M^2}{c^4r^5} - \frac{\cot^2\theta}{r^2}\right] = 0 \qquad (57)$$

This condition can only be satisfied if $r = \infty$. The metric of the geometry being considered is taken, its Christoffel symbols are found, the Riemann curvature tensor is found and through that tensor, both the Ricci curvature tensor and the Ricci scalar are determined. Taking all these components and putting them together yields the Einstein curvature tensor of an object.

CONCLUSION

A non-zero Ricci Scalar based on Howusu Metric Tensor for all gravitational fields in nature equation (54) was formulated, comparing Howusu's Ricci scalar with the well-known Schwarzschild's Ricci scalar it was discovered that they acted alike at $r = \infty$ but different at $r = 0$. The doors are henceforth open to more verification.

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