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Application of the Aboodh Adomian Decomposition Method to Klein-Gordon and Sine-Gordon Equations

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ABSTRACT
In this study, an efficient method is presented for the analysis of the Klein-Gordon (KG) and Sine-Gordon (SG) equations with initial value problems. KG and SG equations are hyperbolic partial differential equations that possess the capability to model phenomena in both quantum and classical mechanics, as well as solitons and condensed matter physics. KG equation represents a relativistic wave equation while SG equation represents the d'Alembert operator with a nonlinear sine term of the dependent variable. The proposed method is based on applying the coupling of Aboodh transformation and Adomian decomposition method (ADM) to partial differential equations and this study is limited to KG and SG equations. The non-linear term is replaced by Adomian polynomials for the index n. The elements of the dependent variable are substituted within the recurrence relation by their respective Aboodh transform components corresponding to the same index. Consequently, the nonlinear problem is addressed in a direct manner, devoid of any linearization or discretization processes. Illustrations are presented to demonstrate the efficacy and veracity of the method. A comparison of the findings with the precise solution indicates that the method proved to be efficient because the results are in closed agreement with the exact solution (errors = 0 with just 5–6 terms). The study concludes that this method can be applied to a variety of linear and nonlinear partial differential equation because Aboodh Adomian Decomposition Method (AADM) provides accurate numerical solutions for linear and nonlinear problems, and can be extended to solve other problems arising in applied science.

Keywords:
Klein-Gordon equation,
Sine-Gordon equation,
Aboodh Transform,
Adomian Decomposition Method.

INTRODUCTION

Differential equations (DE) can model many problems in science and engineering. A simplified mathematical representation of physical reality is known as a mathematical model. Partial Differential Equations are widely used to model nonlinear process that take place in a broad array of scientific fields, including plasma physics, fluid dynamics, solid state physics, mathematical biology, chemical kinematics, etc. (Dehghan & Shokri, 2009) of which Klein-Gordon (KG) and Sine-Gordon (SG) equations are examples.

The KG equation is a crucial mathematical model in both classical and quantum mechanics, recognized as a hyperbolic partial differential equation (PDE) (Yousif & Mahmood, 2017). It has garnered a lot of interest in the study of solitons and condensed matter physics, the recurrence of initial states, in investigating the interaction of solitons in a collision less plasma, and non-linear wave equations (Yusufoglu, 2008). The KG equation is an essential component of mathematical physics as it shows up in fields like quantum field theory, solid state physics, plasma physics, non-linear optics, wave phenomena, mathematical biology, and the study of recurrence of the initial state.

A plane wave equation comparable to the Schrodinger equation called the KG equation predicts the conduct of particles under high energy conditions and velocities close to their celerity (Musielak, 2025). Klein-Gordon equations are widely used in modelling a variety of physical processes like the conduct of the elementary particles alongside the propagation of crystal dislocations.

The following is a representation of the Klein-Gordon equation:

$q_{tt}(\eta,t) - \alpha q_{\eta\eta}(\eta,t) + \beta q(\eta,t) + F(q(\eta,t)) = h(\eta,t)$ (1)

subject to the initial conditions:

$q(\eta,0) = f(\eta),\ q_t(\eta,0) = g(\eta)$ (2)

The function q is dependent on both η and time t; it describes the displacement of a wave at a given position η and time t where α and β are fixed values; $F\big(q(\eta,t)\big)$ is the non-linear function; and $h(\eta,t)$ is a recognized term. If the nonlinear term $F\big(q(\eta,t)\big) = sinq(\eta,t)$ is assigned in (1), the equation turns to a Sine-Gordon equation. To address equation (1), one possible solution involves includes the movement of a quantum scalar or a pseudo-scalar field that consists of particles with zero spins.

The sine of the unknown function and d'Alembert operator are both components of the nonlinear hyperbolic PDE called the Sine-Gordon equation. During research on numerous differential geometry problems, in the nineteenth century, the equation alongside several solution approaches was known. When it was discovered that the equation resulted in solitons in the 1970s, its significance increased significantly. The Sine-Gordon equation is utilized in various corporeal contexts, such as in relativistic field theory, a number of physical applications, including applications in relativistic field theory, Josephson junctions or mechanical transmission lines, and wave propagation in ferromagnetic materials (Maitama & Hamza, 2020).

The equation reads

$q_{tt}(\eta,t) - \alpha q_{\eta\eta}(\eta,t) + \sin q(\eta,t) = 0$ (3)

Subject to the initial conditions:

$q(\eta,0) = f(\eta),\ q_t(\eta,0) = g(\eta)$ (4)

If we consider a mechanical transmission line, $q(x,t)$ is used to indicate the degree of rotation of the pendulums. It is noteworthy that when the amplitude is low, $\sin q \approx$ q.

Eq. (3) simplifies the Klein-Gordon equation

$q_{tt}(\eta,t) - \alpha q_{\eta\eta}(\eta,t) + q(\eta,t) = 0,$ (5)

Admitting solutions in the form

$q(\eta,t) = q_0 \cos(k\eta - \omega t),\ \omega = \sqrt{1+k^2}$ (6)

Here, the area of interest lies in large amplitude solutions of Eq. (3).

Several integral transforms have been developed to address differential equations that involve initial value problems or boundary conditions expressed in the form of integral equations. Integral transform have been widely utilized, leading to a substantial body of work on both the theory and practical application of these transforms. Examples include the Sumudu transform, Fourier transform, Elzaki transform, Hankel transform, Aboodh transform and lots more.

The Aboodh transform was developed via classical Fourier integral and it refers to a valuable tool for analyzing linear DE. Khalid Aboodh introduced the Aboodh Transform in 2013 to simplify the solution procedure for ODEs and PDEs in the temporal domain. It is a convenient mathematical tool for analysing differential equations. It has connections to both the Laplace transform and Elzaki transform. However, the integral transform is unable to analyze non-linear derivatives like KG and SG equations due to the difficulties posed by the non-linear terms. Therefore, we couple Aboodh transform and the Adomian Decomposition Method (ADM) to analyze the KG and SG equations. ADM decomposes the non- linear terms which allow the solution to be obtained as a series that converges quickly.

MATERIALS AND METHODS

Aboodh Transform (Aboodh, 2013)

The Aboodh transform of a function $f(t)$ is defined as

$A\{f(t)\} = \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$ (7)

Table 1: Aboodh Functions and Their Inverse

$K(v)$	$F(t) = A^{-1}\{K(v)\}$
$\frac{1}{s^2}$	1
$\frac{1}{s^3}$	t
$\frac{n!}{s^{n+2}}$	t^n
$\frac{1}{s^2 - as}$	e^{at}
$\frac{a}{s(s^2+a^2)}$	$\sin(at)$
$\frac{1}{(s^2+a^2)}$	$\cos(at)$
$\frac{a}{s(s^2-a^2)}$	$\sinh(at)$
$\frac{1}{(s^2-a^2)}$	$\cosh(at)$

Aboodh Adomian Decomposition Method (Odetunde et al., 2023)

We will briefly discuss how to analyze Klein-Gordon and Sine-Gordon Equations via AADM.

Linear KG Equation

A standard linear KG equation can be expressed as:

$$q_{tt}(\eta,t) - \alpha q_{\eta\eta}(\eta,t) + \beta q(\eta,t) = h(\eta,t) \quad (8)$$

Subject to the initial conditions:

$$q(\eta,0) = f(\eta), \;\; q_t(\eta,0) = g(\eta) \quad (9)$$

Take the Aboodh Transform of Eqn.(8)

$$A\{q_{tt}(\eta,t)\} - \alpha A\{q_{\eta\eta}(\eta,t)\} + \beta A\{q(\eta,t)\} = A\{h(\eta,t)\} \quad (10)$$

$$s^2 A\{q(\eta,t)\} - q(\eta,0) - \frac{q_t(\eta,0)}{s} - \alpha \frac{d^2}{d\eta^2} A\{q(\eta,t)\} + \beta A\{q(\eta,t)\} = A\{h(\eta,t)\}$$

Using the Initial conditions,

$$s^2 A\{q(\eta,t)\} = f(\eta) + \frac{g(\eta)}{s} + \alpha \frac{d^2}{d\eta^2} A\{q(\eta,t)\} - \beta A\{q(\eta,t)\} + A\{h(\eta,t)\}$$

$$A\{q(\eta,t)\} = \frac{f(\eta)}{s^2} + \frac{g(\eta)}{s^3} + \frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q(\eta,t)\} - \frac{\beta}{s^2} A\{q(\eta,t)\} + \frac{1}{s^2} A\{h(\eta,t)\} \quad (11)$$

We consider a series solution below:

$$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t) \quad (12)$$

where $n = 0, 1, 2, \ldots$

Substituting Eqn. (11) into (12)

$$A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = \frac{f(\eta)}{s^2} + \frac{g(\eta)}{s^3} + \frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{\beta}{s^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} + \frac{1}{s^2} A\{h(\eta,t)\}$$

Take the Inverse Aboodh Transform

$$\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = f(\eta) + g(\eta)\, t + t\, A\{h(\eta,t) + A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{\beta}{s^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\}\right] \quad (13)$$

The recursive relation below can be gotten by comparing the two sides of the previous equation $q_0(\eta,t) = f(\eta) + g(\eta)\, t + t\, A\{h(\eta,t)\}$, (14)

$$q_1(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q_0(\eta,t)\} - \frac{\beta}{s^2} A\{q_0(\eta,t)\}\right] \quad (15)$$

$$q_2(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q_1(\eta,t)\} - \frac{\beta}{s^2} A\{q_1(\eta,t)\}\right] \quad (16)$$

$$q_{n+1}(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q_n(\eta,t)\} - \frac{\beta}{s^2} A\{q_n(\eta,t)\}\right] \quad (17)$$

Non-Linear KG Equation

A standard non-linear KG equation can be expressed as:

$$q_{tt}(\eta,t) - \alpha q_{\eta\eta}(\eta,t) + \beta q(\eta,t) + F(q(\eta,t)) = h(\eta,t \quad (18)$$

where $F(q(\eta,t))$ is the non-linear term with IVP:

$$q(\eta,0) = f(\eta), \;\; q_t(\eta,0) = g(\eta) \quad (19)$$

Take the Aboodh Transform of Eqn.(18)

$$A\{q_{tt}(\eta,t)\} - \alpha A\{q_{\eta\eta}(\eta,t)\} + \beta A\{q(\eta,t)\} + A\{F(q(\eta,t))\} = A\{h(\eta,t)\}$$

$$s^2 A\{q(\eta,t)\} - q(\eta,0) - \frac{q_t(\eta,0)}{s} - \alpha \frac{d^2}{d\eta^2} A\{q(\eta,t)\} + \beta A\{q(\eta,t)\} + A\{F(q(\eta,t))\} = A\{h(\eta,t)\}$$

Using the Initial conditions,

$$s^2 A\{q(\eta,t)\} = f(\eta) + \frac{g(\eta)}{s} + \alpha \frac{d^2}{d\eta^2} A\{q(\eta,t)\} - \beta A\{q(\eta,t)\} - A\{F(q(\eta,t))\} + A\{h(\eta,t)\}$$

$$A\{q(\eta,t)\} = \frac{f(\eta)}{s^2} + \frac{g(\eta)}{s^3} + \frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q(\eta,t)\} - \frac{\beta}{s^2} A\{q(\eta,t)\} - \frac{1}{s^2} A\{F(q(\eta,t))\} + \frac{1}{s^2} A\{h(\eta,t)\} \quad (20)$$

The series solution is:

$$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t) \quad (21)$$

The non-linear term is decomposed as:

$$F(q(\eta,t)) = \sum_{n=0}^{\infty} A_n \quad (22)$$

where A_n is called Adomian Polynomials and it can be computed by using:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i q_i(\eta,t)\right)\right]_{\lambda=0} \quad (23)$$

where $n = 0, 1, 2, \ldots$

Substituting Eqn. (21) and (22) into Eqn.(20), we obtain:

$$A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = \frac{f(\eta)}{s^2} + \frac{g(\eta)}{s^3} + \frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{\beta}{s^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{1}{s^2} A\{\sum_{n=0}^{\infty} A_n\} + \frac{1}{s^2} A\{h(\eta,t)\}$$

Take the Inverse Aboodh Transform

$$\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = f(\eta) + g(\eta)\, t + t\, A\{h(\eta,t)\} + A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{\beta}{s^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{1}{s^2} A\{\sum_{n=0}^{\infty} A_n\}\right] \quad (24)$$

The recursive relation below can be gotten by comparing both sides of the previous equation:

$$q_0(\eta,t) = f(\eta) + g(\eta)\, t + t\, A\{h(\eta,t)\} \quad (25)$$

$$q_1(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q_0(\eta,t)\} - \frac{\beta}{s^2} A\{q_0(\eta,t)\} - \frac{1}{s^2} A\{A_0\}\right] \quad (26)$$

$$q_2(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q_1(\eta,t)\} - \frac{\beta}{s^2} A\{q_1(\eta,t)\} - \frac{1}{s^2} A\{A_1\}\right] \quad (27)$$

$$q_{n+1}(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2} A\{q_n(\eta,t)\} - \frac{\beta}{s^2} A\{q_n(\eta,t)\} - \frac{1}{s^2} A\{A_n\}\right] \quad (28)$$

Sine-Gordon Equation

The standard nonlinear SG equation is:

$$q_{tt}(\eta,t) - \alpha q_{\eta\eta}(\eta,t) - \beta \sin(q(\eta,t)) = 0, \quad (29)$$

with IVP:

$$q(\eta,0) = f(\eta), \;\; q_t(\eta,0) = g(\eta) \quad (30)$$

Take the Aboodh Transform of Eqn.(29),

$$A\{q_{tt}(\eta,t)\} - \alpha A\{q_{\eta\eta}(\eta,t)\} - \beta A\{\sin(q(\eta,t)\} = 0$$

$$s^2 A\{q(\eta,t)\} - q(\eta,0) - \frac{q_t(\eta,0)}{s} - \alpha \frac{d^2}{d\eta^2} A\{q(\eta,t)\} - A\{\beta \sin(q(\eta,t)\} = 0$$

Using the initial conditions in Eqn. (30)

$$s^2 A\{q(\eta,t)\} = f(\eta) + \frac{g(\eta)}{s} + \alpha \frac{d^2}{d\eta^2} A\{q(\eta,t)\} + \beta A\{\sin(q(\eta,t)\}$$

$A\{q(\eta,t)\} = \frac{f(\eta)}{s^2} + \frac{g(\eta)}{s^3} + \frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{q(\eta,t)\} + \frac{\beta}{s^2}A\{\sin(q(\eta,t)\}$ (31)

We consider the series solution:

$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t)$ (32)

The non-linear term is decomposed as:

$\sin\big(q(\eta,t)\big) = \sum_{n=0}^{\infty} A_n$ (33)

where A_n is called Adomian Polynomials and it can be computed by using:

$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i\, q_i(\eta,t)\right)\right]_{\lambda=0}$ (34)

where $n = 0, 1, 2, \dots$

Substituting Eqn. (33) and (33) into Eqn.(31), we obtain:

$A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = \frac{f(\eta)}{s^2} + \frac{g(\eta)}{s^3} + \frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} + \frac{\beta}{s^2}A\{\sum_{n=0}^{\infty} A_n\}$

Take the Inverse Aboodh Transform

$\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = f(\eta) + g(\eta)\, t + A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} + \frac{\beta}{s^2}A\{\sum_{n=0}^{\infty} A_n\}\right]$ (35)

Comparing both sides of the equation above, the following recursive relation can be generated:

$q_0(\eta,t) = f(\eta) + g(\eta)\, t,$ (36)

$q_1(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{q_0(\eta,t)\} + \frac{\beta}{s^2}A\{A_0\}\right]$ (37)

$q_2(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{q_1(\eta,t)\} + \frac{\beta}{s^2}A\{A_1\}\right]$ (38)

$q_{n+1}(\eta,t) = A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{q_n(\eta,t)\} + \frac{\beta}{s^2}A\{A_n\}\right]$ (39)

RESULTS AND DISCUSSION

Example 1

Given the linear Klein-Gordon equation

$q_{tt}(\eta,t) - q_{\eta\eta}(\eta,t) + q(\eta,t) = 2\sin\eta$ (40)

Subject to the initial conditions:

$q(\eta,0) = \sin\eta,\quad q_t(\eta,0) = 1$ (41)

Exact solution: $q(\eta,t) = \sin\eta + \sin t$

Take the Aboodh Transform of Eqn.(40)

$A\{q_{tt}(\eta,t)\} - A\{q_{\eta\eta}(\eta,t)\} + A\{q(\eta,t)\} = 2A\{\sin\eta)\}$

$s^2A\{q(\eta,t)\} - q(\eta,0) - \frac{q_t(\eta,0)}{s} - \frac{d^2}{d\eta^2}A\{q(\eta,t)\} + A\{q(\eta,t)\} = \frac{2}{s^2}\sin\eta$ (42)

Using the Initial conditions,

$s^2A\{q(\eta,t)\} = \sin\eta + \frac{1}{s} + \frac{d^2}{d\eta^2}A\{q(\eta,t)\} - A\{q(\eta,t)\} + \frac{2}{s^2}\sin\eta$

$A\{q(\eta,t)\} = \frac{\sin\eta}{s^2} + \frac{1}{s^3} + \frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q(\eta,t)\} - \frac{1}{s^2}A\{q(\eta,t)\} + \frac{2}{s^4}A\{\sin\eta\}$ (43)

The series solution is:

$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t)$ (44)

where $n = 0, 1, 2, \dots$

Substituting Eqn. (43) into Eqn.(44), we obtain:

$A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = \frac{\sin\eta}{s^2} + \frac{1}{s^3} + \frac{1}{s^2}\frac{d^2}{d\eta^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{1}{s^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} + \frac{2}{s^4}\sin\eta$ (45)

Take the Inverse Aboodh Transform

$\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = \sin\eta + t + t^2\sin\eta + A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{1}{s^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\}\right]$ (46)

Comparing both sides of the equation above, the following recursive relation can be generated:

$q_0(\eta,t) = \sin\eta + t + t^2\sin\eta,$ (47)

$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_0(\eta,t)\} - \frac{1}{s^2}A\{q_0(\eta,t)\}\right]$

$q_1(\eta,t) = A^{-1}\left[-\frac{\sin\eta}{s^4} - \frac{2\sin\eta}{s^6} - \left(\frac{\sin\eta}{s^4} + \frac{1}{s^5} + \frac{2\sin\eta}{s^6}\right)\right]$

$q_1(\eta,t) = A^{-1}\left[-\frac{2\sin\eta}{s^4} - \frac{1}{s^5} - \frac{4\sin\eta}{s^6}\right]$

$q_1(\eta,t) = -t^2\sin\eta - \frac{t^3}{3!} - \frac{t^4\sin\eta}{3!}$ (48)

$q_2(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_1(\eta,t)\} - \frac{1}{s^2}A\{q_1(\eta,t)\}\right]$

$q_2(\eta,t) = A^{-1}\left[\frac{2\sin\eta}{s^6} + \frac{4\sin\eta}{s^8} - \left(-\frac{2\sin\eta}{s^6} - \frac{1}{s^7} - \frac{4\sin\eta}{s^8}\right)\right]$

$q_2(\eta,t) = A^{-1}\left[\frac{4\sin\eta}{s^6} + \frac{1}{s^7} + \frac{8\sin\eta}{s^8}\right]$

$q_2(\eta,t) = \frac{t^4\sin\eta}{3!} + \frac{t^5}{5!} + \frac{8\,t^6\sin\eta}{6!}$ (49)

$q_3(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_2(\eta,t)\} - \frac{1}{s^2}A\{q_2(\eta,t)\}\right]$

$q_3(\eta,t) = A^{-1}\left[-\frac{4\sin\eta}{s^8} - \frac{8\sin\eta}{s^{10}} - \left(\frac{4\sin\eta}{s^8} + \frac{1}{s^9} + \frac{8\sin\eta}{s^{10}}\right)\right]$

$q_3(\eta,t) = A^{-1}\left[-\frac{8\sin\eta}{s^8} - \frac{1}{s^9} - \frac{16\sin\eta}{s^{10}}\right]$

$q_3(\eta,t) = -\frac{8\,t^6\sin\eta}{6!} - \frac{t^7}{7!} - \frac{16\,t^8 sin\eta}{8!}$ (50)

$q_4(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_3(\eta,t)\} - \frac{1}{s^2}A\{q_3(\eta,t)\}\right]$

$q_4(\eta,t) = A^{-1}\left[\frac{8\sin\eta}{s^{10}} + \frac{16\sin\eta}{s^{12}} - \left(-\frac{8\sin\eta}{s^{10}} - \frac{1}{s^{11}} - \frac{16\sin\eta}{s^{12}}\right)\right]$

$q_4(\eta,t) = A^{-1}\left[\frac{16\sin\eta}{s^{10}} + \frac{1}{s^{11}} + \frac{32\sin\eta}{s^{12}}\right]$

$q_4(\eta,t) = \frac{16\,t^8\sin\eta}{8!} + \frac{t^9}{9!} + \frac{32\,t^{10} sin\eta}{10!}$ (51)

The series solution is given by:

$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t)$

$= q_0(\eta,t) + q_1(\eta,t) + q_2(\eta,t) + q_3(\eta,t) + .$

Thus,

$q(\eta,t) = \sin\eta + t + t^2\sin\eta - t^2\sin\eta - \frac{t^3}{3!} - \frac{t^4\sin\eta}{3!} + \frac{t^4\sin\eta}{3!} + \frac{t^5}{5!} + \frac{8\,t^6\sin\eta}{6!} - \frac{8\,t^6\sin\eta}{6!} - \frac{t^7}{7!} - \frac{16\,t^8\sin\eta}{8!} + \frac{16\,t^8\sin\eta}{8!} + \frac{t^9}{9!} + \frac{32\,t^{10}\sin\eta}{10!}$

$q(\eta,t) = \sin\eta + t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots\dots$ (52)

Since $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots\dots,$

$q(\eta,t) = \sin\eta + \sin t$

Table 2: Comparison of the Exact and Approximate Solution at $0 \le \eta \le 1, t = 0.3$

η	Exact	AADM	Error
0	0	0	0
0.1	0.3953536233	0.3953536233	0.0000000000
0.2	0.4941895375	0.4941895375	0.0000000000
0.3	0.5910404133	0.5910404133	0.0000000000
0.4	0.684938549	0.684938549	0.000000000
0.5	0.7749457453	0.7749457453	0.0000000000
0.6	0.8601626801	0.8601626801	0.0000000000
0.7	0.9397378939	0.9397378939	0.0000000000
0.8	1.012876298	1.012876298	0.000000000
0.9	1.078847116	1.078847116	0.000000000
1.0	1.136991191	1.136991191	0.000000000

Example 2

Given the nonlinear Klein-Gordon equation

$$q_{tt}(\eta,t) - q_{\eta\eta}(\eta,t) + q^2(\eta,t) = \eta^2 t^2 \qquad (53)$$

Subject to the initial conditions

$$q(\eta,0) = 0,\ \ q_t(\eta,0) = \eta \qquad (54)$$

Exact solution: $q(\eta) = \eta t$

Take the Aboodh Transform of Eqn.(53)

$$A\{q_{tt}(\eta,t)\} - A\{q_{\eta\eta}(\eta,t)\} + A\{q(\eta,t)\} + A\{F(q(\eta,t))\} = A\{\eta^2 t^2\}$$

$$s^2 A\{q(\eta,t)\} - q(\eta,0) - \frac{q_t(\eta,0)}{s} - \frac{d^2}{d\eta^2} A\{q(\eta,t)\} + A\{F(q(\eta,t))\} = \frac{2\eta^2}{s^4} \qquad (55)$$

Using the Initial conditions,

$$s^2 A\{q(\eta,t)\} - \frac{\eta}{s} - \frac{d^2}{d\eta^2} A\{q(\eta,t)\} + A\{F(q(\eta,t))\} = \frac{2\eta^2}{s^4}$$

$$A\{q(\eta,t)\} = \frac{\eta}{s^3} + \frac{2\eta^2}{s^6} + \frac{1}{s^2}\frac{d^2}{d\eta^2} A\{q(\eta,t)\} - \frac{1}{s^2} A\{F\big(q(\eta,t)\big)\} \qquad (56)$$

The series solution is:

$$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t) \qquad (57)$$

The non-linear term is decomposed as:

$$F(q(\eta,t)) = \sum_{n=0}^{\infty} A_n \qquad (58)$$

where A_n is called Adomian Polynomials and it can be computed by using:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i q_i(\eta,t)\right)\right]_{\lambda=0} \qquad (59)$$

where $n = 0, 1, 2, \ldots$

Substituting Eqn. (56) and (58) into Eqn.(57), we obtain:

$$A\{\sum_{n=0}^{\infty} q_n(j,t)\} = \frac{j}{s^3} + \frac{2j^2}{s^6} + \frac{1}{s^2}\frac{d^2}{d\eta^2} A\{\sum_{n=0}^{\infty} q_n(j,t)\} - \frac{1}{s^2} A\{A_n\}$$

Take the Inverse Aboodh Transform

$$\{\sum_{n=0}^{\infty} q_n(\eta,t)\} = \eta t + \frac{\eta^2 t^4}{12} + A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2} A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} - \frac{1}{s^2} A\{\sum_{n=0}^{\infty} A_n\}\right] \qquad (60)$$

The recursive relation below can be gotten by comparing the two sides of the previous equation

$$q_0(\eta,t) = \eta t + \frac{\eta^2 t^4}{12} \qquad (61)$$

$$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2} A\{q_0(\eta,t)\} - \frac{1}{s^2} A\{A_0\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{4}{s^8} - \frac{1}{s^2} A\{q_0^2\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{4}{s^8} - \frac{1}{s^2} A\left\{\eta^2 t^2 + \frac{\eta^3 t^5}{6} + \frac{\eta^4 t^8}{144}\right\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{4}{s^8} - \frac{2\eta^2}{s^6} - \frac{20\eta^3}{s^9} - \frac{280\eta^4}{s^{12}}\right]$$

$$q_1(\eta,t) = \left[\frac{4t^6}{6!} - \frac{2\eta^2 t^4}{4!} - \frac{20\eta^3 t^7}{7!} - \frac{280\eta^4 t^{10}}{s^{12}}\right]$$

$$q_1(\eta,t) = \left[\frac{t^6}{180} - \frac{\eta^2 t^4}{12} - \frac{\eta^3 t^7}{252} - \frac{\eta^4 t^{10}}{12960}\right] \qquad (62)$$

$$q_2(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2} A\{q_1(\eta,t)\} - \frac{1}{s^2} A\{A_1\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[-\frac{4}{s^8} - \frac{120\eta}{s^{11}} - \frac{3360\eta^2}{s^{14}} - \frac{1}{s^2} A\{2q_0q_1\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[-\frac{4}{s^8} - \frac{120\eta}{s^{11}} - \frac{3360\eta^2}{s^{14}} - \frac{2}{s^2} A\left\{\frac{\eta t^7}{180} - \frac{\eta^3 t^5}{12} - \frac{\eta^4 t^8}{252} - \frac{\eta^5 t^{11}}{12960} + \frac{\eta^2 t^{10}}{2160} - \frac{\eta^4 t^8}{144} - \frac{\eta^5 t^{11}}{3024} - \frac{\eta^6 t^{14}}{155520}\right\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[-\frac{4}{s^8} - \frac{120\eta}{s^{11}} - \frac{3360\eta^2}{s^{14}} - \frac{2}{s^2}\left\{\frac{7!\,\eta}{180 s^9} - \frac{5!\eta^3}{12\, s^7} - \frac{8!\eta^4}{252\, s^{10}} - \frac{11!\eta^5}{12960\, s^{13}} + \frac{10!\eta^2}{2160\, s^{12}} - \frac{8!\eta^4}{144\, s^{10}} - \frac{11!\eta^5}{3024\, s^{13}} - \frac{14!\eta^6}{155520\, s^{16}}\right\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[-\frac{4}{s^8} - \frac{120\eta}{s^{11}} - \frac{3360\eta^2}{s^{14}} - \frac{56\eta}{s^{11}} + \frac{20\eta^3}{s^9} + \frac{320\eta^4}{s^{12}} + \frac{6160\eta^5}{s^{15}} - \frac{3360\eta^2}{s^{14}} + \frac{560\eta^4}{s^{12}} + \frac{26400\eta^5}{s^{15}} + \frac{1121120\eta^6}{s^{18}}\right]$$

$$q_2(\eta,t) = A^{-1}\left[-\frac{4}{s^8} + \frac{20\eta^3}{s^9} - \frac{176\eta}{s^{11}} + \frac{880\eta^4}{s^{12}} - \frac{6720\eta^2}{s^{14}} + \frac{32560\eta^5}{s^{15}} + \frac{1121120\eta^6}{s^{18}}\right]$$

$$q_2(\eta,t) = -\frac{4t^6}{6!} + \frac{20\eta^3 t^7}{7!} - \frac{176\eta t^9}{9!} + \frac{880\eta^4 t^{10}}{10!} - \frac{6720\eta^2 t^{12}}{12!} + \frac{32560\eta^5 t^{13}}{13!} + \frac{1121120\eta^8 t^{16}}{16!}$$

$$q_2(\eta,t) = -\frac{t^6}{180} + \frac{\eta^3 t^7}{252} - \frac{11\eta t^9}{22680} + \frac{\eta^4 t^{10}}{12960} - \frac{\eta^2 t^{12}}{71280} + \frac{37\eta^5 t^{13}}{7076160} + \frac{\eta^8 t^{16}}{18662400} \qquad (63)$$

The series solution is:

$$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t)$$
$$= q_0(\eta,t) + q_1(\eta,t) + q_2(\eta,t) + \cdots \ldots$$

Thus,

$$q(\eta,t) = \eta t + \frac{\eta^2 t^4}{12} + \frac{t^6}{180} - \frac{\eta^2 t^4}{12} - \frac{\eta^3 t^7}{252} - \frac{\eta^4 t^{10}}{12960} - \frac{t^6}{180} + \frac{\eta^3 t^7}{252} - \frac{11\eta t^9}{22680} + \frac{11\eta^4 t^{10}}{45360} - \frac{\eta^2 t^{12}}{71280} + \frac{37\eta^5 t^{13}}{7076160} + \frac{\eta^8 t^{16}}{18662400} \qquad (64)$$

Here, we have noise terms which are

$\frac{\eta^2 t^4}{12}, \frac{t^6}{180}, \frac{\eta^3 t^7}{252}, \frac{\eta^4 t^{10}}{12960}, \frac{11\eta t^9}{22680}, \frac{\eta^2 t^{12}}{71280}, \frac{37\eta^5 t^{13}}{7076160}, \frac{\eta^8 t^{16}}{18662400}$ *and … so on*

$$\therefore q(\eta,t) = \eta t$$

Table 3: Comparison of the Exact and Approximate Solution At $0 \leq \eta \leq 1, t = 0.3$

η	Exact	AADM	Error
0	0	0	0
0.1	0.03	0.03	0.00
0.2	0.06	0.06	0.00
0.3	0.09	0.09	0.00
0.4	0.12	0.12	0.00
0.5	0.15	0.15	0.00
0.6	0.18	0.18	0.00
0.7	0.21	0.21	0.00
0.8	0.24	0.24	0.00
0.9	0.27	0.27	0.00
1.0	0.30	0.30	0.00

Example 3

Given the nonlinear Sine-Gordon equation

$$q_{tt}(\eta,t) - q_{\eta\eta}(\eta,t) - \sin\big(q(\eta,t)\big) = 0, \quad (65)$$

subject to the initial conditions:

$$q(\eta,0) = \frac{\pi}{2}, \quad q_t(\eta,0) = 0 \quad (66)$$

Take the Aboodh Transform of Eqn.(65),

$$A\{q_{tt}(\eta,t)\} - A\{q_{\eta\eta}(\eta,t)\} - A\{\sin(q(\eta,t)\} = 0$$

$$s^2A\{q(\eta,t)\} - q(\eta,0) - \frac{q_t(\eta,0)}{s} - \frac{d^2}{d\eta^2}A\{q(\eta,t)\} - A\{\sin(q(\eta,t)\} = 0$$

Using the initial conditions in Eqn. (66)

$$s^2A\{q(\eta,t)\} = \frac{\pi}{2} + \frac{d^2}{d\eta^2}A\{q(\eta,t)\} + A\{\sin(q(\eta,t)\}$$

$$A\{q(\eta,t)\} = \frac{\pi}{2s^2} + \frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q(\eta,t)\} + \frac{1}{s^2}A\{\sin(q(\eta,t)\} \quad (67)$$

We consider the series solution:

$$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t) \quad (68)$$

The non-linear term is decomposed as:

$$\sin\big(q(\eta,t)\big) = \sum_{n=0}^{\infty} A_n \quad (69)$$

where A_n is called Adomian Polynomials and it can be computed by using:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i q_i(\eta,t)\right)\right]_{\lambda=0} \quad (70)$$

where $n = 0, 1, 2, \ldots$

Substituting Eqn. (67) and (69) into Eqn.(68), we obtain:

$$A\{\textstyle\sum_{n=0}^{\infty} q_n(\eta,t)\} = \frac{\pi}{2s^2} + \frac{1}{s^2}\frac{d^2}{d\eta^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} + \frac{1}{s^2}A\{\sum_{n=0}^{\infty} A_n\} \quad (71)$$

Take the Inverse Aboodh Transform

$$\{\textstyle\sum_{n=0}^{\infty} q_n(\eta,t)\} = \frac{\pi}{2} + A^{-1}\left[\frac{\alpha}{s^2}\frac{d^2}{d\eta^2}A\{\sum_{n=0}^{\infty} q_n(\eta,t)\} + \frac{\beta}{s^2}A\{\sum_{n=0}^{\infty} A_n\}\right] \quad (72)$$

The recursive relation below can be gotten by comparing the two sides of the previous equation:

$$q_0(\eta,t) = \frac{\pi}{2}, \quad (73)$$

$$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_0(\eta,t)\} + \frac{1}{s^2}A\{A_0\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}\frac{\pi}{2s^2} + \frac{1}{s^2}A\{\sin q_0(\eta,t)\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{\sin q_0(\eta,t)\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{\sin\frac{\pi}{2}\}\right]$$

$$q_1(\eta,t) = A^{-1}\left[\frac{1}{s^2}.\frac{1}{s^2}\right]$$

$$q_1(\eta,t) = \frac{t^2}{2} \quad (74)$$

$$q_2(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_1(\eta,t)\} + \frac{1}{s^2}A\{A_1\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}\frac{1}{s^4} + \frac{1}{s^2}A\{q_1(x,t)\cos q_0(\eta,t)\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{q_1(\eta,t)\cos q_0(\eta,t)\}\right]$$

$$q_2(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{\frac{1}{s^4}\cos\frac{\pi}{2}\}\right]$$

$$q_2(\eta,t) = 0 \quad (75)$$

$$q_3(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_2(\eta,t)\} + \frac{1}{s^2}A\{A_2\}\right]$$

$$q_3(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_2(\eta,t)\} + \frac{1}{s^2}A\{A_2\}\right]$$

$$q_3(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{A_2\}\right]$$

$$q_3(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{q_2(\eta,t)\cos(q_0(\eta,t)) - \frac{1}{2!}q_1^2(\eta,t)\sin q_0(\eta,t)\}\right]$$

$$q_3(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{-\frac{t^4}{8}\}\right]$$

$$q_3(\eta,t) = -\frac{1}{8}A^{-1}\left[\frac{1}{s^2}A\{t^4\}\right]$$

$$q_3(\eta,t) = -\frac{1}{8}A^{-1}\left[\frac{4!}{s^8}\right]$$

$$q_3(\eta,t) = -\frac{4!}{8}.\frac{t^6}{6!}$$

$$q_3(\eta,t) = -\frac{t^6}{240} \quad (76)$$

$$q_4(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_3(\eta,t)\} + \frac{1}{s^2}A\{A_3\}\right]$$

$$q_4(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{q_3(\eta,t)\cos(q_0(\eta,t)) - q_1(\eta,t)q_2(\eta,t)\sin q_0(\eta,t\} - \frac{1}{3!}q_1^3(\eta,t)\cos q_0(\eta,t\}\right]$$

$$q_4(\eta,t) = 0 \quad (77)$$

$$q_5(\eta,t) = A^{-1}\left[\frac{1}{s^2}\frac{d^2}{d\eta^2}A\{q_4(\eta,t)\} + \frac{1}{s^2}A\{A_4\}\right]$$

$q_5(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{q_4(\eta,t)cos(q_0(\eta,t)) - \left(q_1(\eta,t)q_3(\eta,t) + \frac{q_2^2(\eta,t)}{2!}\right)sin(q_0(\eta,t)) - \frac{q_1^2 q_2}{2!}\ cos\ q_0(\eta,t) + \frac{q_1^4}{4!}sin(q_0(\eta,t))\}\right]$

$q_5(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{\frac{t^8}{280} + \frac{t^{16}}{24}\}\right]$

$$q_5(\eta,t) = A^{-1}\left[\frac{1}{s^2}A\{\frac{t^8}{280} + \frac{t^{16}}{24}\}\right]$$

$$q_5(\eta,t) = \frac{83}{13440}A^{-1}\left[\frac{1}{s^2}A\{t^8\}\right]$$

$$q_5(\eta,t) = \frac{83}{13440}A^{-1}\left[\frac{8}{s^{12}}\right]$$

$$q_5(\eta,t) = \frac{83}{1680}.\left[\frac{t^{10}}{10!}\right] \quad (78)$$

The series solution is given by:

$q(\eta,t) = \sum_{n=0}^{\infty} q_n(\eta,t)$

$= q_0(\eta,t) + q_1(\eta,t) + q_2(\eta,t) + q_3(\eta,t) + \cdots ...$

Thus,

$q(\eta,t) = \frac{\pi}{2} + \frac{t^2}{2} - \frac{t^6}{240} + \frac{t^{10}}{34560} + \cdots ...$

The series completely agrees with the outcome gotten by Mataima & Hamza (2020)

Discussion

In this study, the Aboodh–Adomian Decomposition Method (AADM) was successfully applied to obtain approximate solutions to both the KG and SG equations. The method combines the advantages of the Aboodh transform and the Adomian Decomposition Method to effectively handle both linear and nonlinear partial differential equations.

For the linear and nonlinear forms of the KG equation, the AADM was implemented using different sets of parameters and initial conditions. The solutions obtained through this method were compared with the corresponding exact analytical solutions. The numerical results presented in the tables show that the approximate solutions generated by the AADM are in excellent agreement with the exact solutions. This demonstrates that the method provides highly accurate approximations while maintaining computational simplicity.

Similarly, the AADM was applied to the nonlinear SG equation. The approximate solution obtained from the present method was compared with results previously reported in the literature. The comparison indicates that the results obtained using the AADM closely match those reported by earlier researchers, thereby confirming the validity and reliability of the method for solving nonlinear wave equations.

CONCLUSION

This study successfully applied the Aboodh Adomian Decomposition Method (AADM) to obtain approximate solutions for both the KG and SG equations. The nonlinear terms were handled effectively using Adomian polynomials, while the Aboodh transform simplified the differential equations into solvable recursive relations. The method avoids linearization and discretization, thereby preserving the original structure of the equations. Numerical results demonstrated excellent agreement with exact solutions and previously published results, confirming the accuracy and reliability of the method. Consequently, AADM provides an effective analytical tool for solving a wide class of linear and nonlinear partial differential equations arising in applied mathematics and physics.

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