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L-stable Implicit Two-Step Modified Extended Hybrid Block Backward Differentiation Formulae for Solving Nonlinear Second-Order Differential-Algebraic Equations of Higher Indexes

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ABSTRACT
Differential-algebraic equations (DAEs) serve as essential extensions of traditional differential equations, incorporating algebraic constraints that are critical for modelling complex mechanical systems and electrical circuits. While numerical solutions for low-index DAEs are well-established, solving higher-order nonlinear systems remains a challenge. This paper presents a two-step L-stable Implicit Modified Extended Hybrid Block Backward Differentiation Formula (IMEHBBDF) specifically designed for Second-Order Nonlinear Differential Algebraic Equations (SONDAEs) of indexes one, two and three. Differential-algebraic equations have been transformed to ordinary differential equations through differentiation-index method. The algorithm was derived using a collocation and interpolation technique to construct an implicit second-derivative formula. This approach generates a unified block of methods that provides simultaneous approximate solutions for the system. Theoretical analysis demonstrates that the IMEHBBDF was consistent, convergent, and achieves an exceptionally high order of fourteen. The efficiency, accuracy and adaptability of the proposed method are validated through the numerical solution of four diverse examples, demonstrating its superiority in handling complex nonlinear DAEs.

Keywords:
Backward Differentiation Formulae,
Differential-Algebraic Equations,
Implicit Modified Extended Hybrid Block,
Rate of Convergence,
Stability Analysis.

INTRODUCTION

Differential-algebraic equations (DAEs) are a combination of differential and algebraic equations that are utilized in various branches of science and engineering. These include mechanical or multibody systems, chemical processes, optimal control, electric circuit design, dynamical systems, electrical networks, and circuit analysis. Several authors have developed numerical methods to obtain solutions to the DAEs of the form (1). Akinfenwa et. al, 2014; Akinnukawe et. al, 2018 through Linear Multistep Methods.

$$F\{t, x(t), x'(t), x''(t)\} \quad (1)$$

Using semi-analytical methods such as Padé's approximations (Ascher & Petzold, 1999), Adomian decomposition Methods (ADMs) (Biazar, Babolian, & Islam, 2003; Biazar, Tango, Babolian, & Islam, 2004; Donalchili & Jafari, 2020), Variational Iterative Methods (VIMs), Pseudospectral Methods (Vazquez-Leal1 and Sarmiento-Reyes, 2015); Differential Transform Methods (DTM) (Brahim Benhammouda, 2023); Variational Formulation Techniques (VFT) (Radhi Ali and Ghazwa Faisal, 2015; Radhi A. Zaboon, Ghazwa F. Abd, 2021). Block method which preserves the self-starting characteristics of the Runge-Kutta method offers the advantage of approximating the solution of (1) at multiple points (Akinfenwa & Okunuga, 2014).

Motivated by the work of Akinnukawe et al. (2019), where they proposed block hybrid methods for DAEs, this paper presents an Implicit Modified Extended Hybrid Block Second Derivative Backward Differentiation Formulae that is not only self-starting but also possesses good accuracy, stability properties, while being computationally efficient in terms of the number of functions evaluation per step, for effective and efficient solution of the differential-algebraic equations (1).

MATERIALS AND METHODS

Construction of Implicit Modified Extended Hybrid Block Backward Differentiation Formulae

A continuous formulation for both the main and supplementary methods is established and applied to derive the discrete implicit modified extended hybrid block backward differentiation formulae (IMEHBBDF) over the interval $[\, t_n \,,\, t_{n+3}]$. In the development of the IMEHBBDF, it is presumed that $Y(t)$ serves as a local representation of the exact solution $y(t)$ defined by,

$$Y(t) = \sum_{j=0}^{p+q-1} w_j \vartheta_j(t) \qquad (2)$$

Where w_j are unknown coefficients to be determined, and $\vartheta_j(t)$ are polynomial basis function of degree $p + q - 1$ such that the number of interpolation points p and the number of distinct collocation points q are respectively chosen to satisfy $1 \le p \le s, s = 3$ and$q > 0$. The proposed class of methods is thus developed by imposing the following conditions

$$\sum_{j=0}^{4k+6} w_j\, t_{n+i}^j = y_{n+i} \;,\; i = 0\left(\tfrac{1}{2}\right)2 \qquad (3)$$

$$\sum_{j=0}^{4k+6} w_j\, t_{n+i}^j = f_{n+i} \;,\; i = 0\left(\tfrac{1}{2}\right)3 \qquad (4)$$

$$\sum_{j=0}^{4k+6} w_j\, t_{n+i}^j = g_{n+i} \;,\; i = \tfrac{3}{2}, 2, 3 \qquad (5)$$

Equations from (3) to (5) leads to a system of fifteen equations which is solved to obtain thew_j's whose values are substituted into (2) to give $Y(x) = \sum_{j=0}^{4}\left(\alpha_{\frac{j}{2}}(x) y_{n+\frac{j}{2}}\right) + h\left(\sum_{i=0}^{6}\beta_{\frac{i}{2}}(x) f_{n+\frac{i}{2}}\right) + h^2\left(\gamma_{\frac{\mu}{2}}(x) g_{n+\frac{\mu}{2}} + \gamma_k(x) g_{n+k} + \gamma_{k+1}(x) g_{n+k+1}\right)$ (6)

Where $\alpha_{\frac{j}{2}}(x),\ j = 0(1)4;\ \beta_{\frac{i}{2}}(x)\,, ij = 0(1)6,\ \gamma_k(x)\,,$ *and* $\gamma_{\frac{\mu}{2}}(x)$ are continuous coefficients, $\mu = 3$ and k =2. is step number. The modified extended hybrid block method (7) is subsequently obtained by evaluating IMEHBBDF (6) at points $x = x_{n+k+1}$ and $x = x_{n+k+\frac{1}{2}}$ to obtain the main discrete method given by (7).

$$\left.\begin{aligned}
y_{n+3} = {} & \frac{h^2}{1008989}\left\{9420000 g_{n+\frac{3}{2}} - 3138750 g_{n+2} - 9150 g_{n+3}\right\} + \frac{h}{1008989}\Big\{-730080 f_{n+\frac{1}{2}} + \\
& 27200000 f_{n+\frac{3}{2}} + 972000 f_{n+\frac{5}{2}} - 10750 f_n - 12453750 f_{n+1} + 39555000 f_{n+2} + 182240 f_{n+3}\Big\} \\
& - \frac{179803125}{10008989} y_{n+2} - \frac{4859136}{10008989} y_{n+\frac{1}{2}} + \frac{249120000}{10008989} y_{n+\frac{3}{2}} - \frac{116875}{10008989} y_n - \frac{63331875}{10008989} y_{n+1} \\
y_{n+\frac{5}{2}} = {} & \frac{h^2}{871766496}\left\{-699684000 g_{n+\frac{3}{2}} + 370352250 g_{n+2} + 103200 g_{n+3}\right\} + \frac{h}{871766496}\Big\{43962480 f_{n+\frac{1}{2}} - \\
& 2688656000 f_{n+\frac{3}{2}} + 96221520 f_{n+\frac{5}{2}} - 1062980 f_{n+3} - 617470 f_n + 810364500 f_{n+1} - 3349039500 f_{n+2}\Big\} \\
& + \frac{623023125}{32287648} y_{n+2} + \frac{344625}{1008989} y_{n+\frac{1}{2}} - \frac{641921500}{27242703} y_{n+\frac{3}{2}} + \frac{6760121}{871766496} y_n + \frac{9923875}{2017978} y_{n+1}
\end{aligned}\right\} \qquad (7)$$

To obtain the additional method, we differentiate continuous form (CF) (6) twice with respect to x, then we obtain (8)

$$y''(x) = \frac{1}{h^2}\left\{\sum_{j=0}^{4}\left(\alpha''_{\frac{j}{2}}(x) y_{n+\frac{j}{2}}\right) + h\left(\sum_{i=0}^{6}\beta''_{\frac{i}{2}}(x) f_{n+\frac{i}{2}}\right) + h^2\left(\gamma''_{\frac{\mu}{2}}(x) g_{n+\frac{\mu}{2}} + \gamma''_k(x) g_{n+k} + \gamma''_{k+1}(x) g_{n+k+1}\right)\right\} \qquad (8)$$

By collocating (9) at some specific point steps$x_{n+\frac{i}{2}}$, where$i = 0, 1, 2, 5$, the complementary method is given as (9)

$$\left.\begin{aligned}
h^2 g_{n+\frac{1}{2}} = {} & \frac{h^2}{8717664960}\left\{130423092000 g_{n+\frac{3}{2}} - 20200907250 g_{n+2} + 5881440 g_{n+3}\right\} - \frac{5617675}{4035956} y_{n+1} + \frac{h}{8717664960} \\
& \{-108837375408 f_{n+\frac{1}{2}} - 532160000 f_{n+\frac{3}{2}} + 9829600 f_{n+\frac{5}{2}} - 538292 f_{n+3} - 619746650 f_n - 4723150905 \\
& 00 f_{n+1} - 31001500 f_{n+2}\} + \frac{353826375}{193725888} y_{n+2} - \frac{181064150}{3026967} y_{n+\frac{1}{2}} - \frac{10372550}{27242703} y_{n+\frac{3}{2}} + \frac{1552906175}{1743532992} y_n \\
h^2 g_{n+\frac{5}{2}} = {} & \frac{h^2}{26152994880}\left\{173606435200 g_{n+\frac{3}{2}} - 7231714050 g_{n+2} - 399593760 g_{n+3}\right\} + \frac{h}{26152994880} \\
& \{-120553737 f_{n+\frac{1}{2}} + 593684200 f_{n+\frac{3}{2}} + 175707432 f_{n+\frac{5}{2}} + 42848164 f_{n+3} - 17252782 f_n - 2151910100 f_{n+1}\} \\
& + 811451100 f_{n+2} + \frac{90830475}{64575296} y_{n+2} - \frac{31280990}{1008989} y_{n+\frac{1}{2}} + \frac{15254670}{81728109} y_{n+\frac{3}{2}} - \frac{3767337805}{5230598976} y_n - \frac{17279765}{4035956} y_{n+1} \\
g_{n+1} = {} & \frac{h^2}{681067575}\left\{1505942400 g_{n+\frac{3}{2}} - 175444650 g_{n+2} - 41865 g_{n+3}\right\} + \frac{h}{681067575}\Big\{201036096 f_{n+\frac{1}{2}} \\
& - 3617868800 f_{n+\frac{3}{2}} + 7462080 f_{n+\frac{5}{2}} + 1785970 f_n - 6225216075 f_{n+1} + 2826137700 f_{n+2} - 387221 f_{n+3}\Big\} \\
& - \frac{21963161}{1008989} y_{n+2} + \frac{2455040}{1008989} y_{n+\frac{1}{2}} + \frac{1798508032}{27242703} y_{n+\frac{3}{2}} + \frac{823619}{27242703} y_n - \frac{47133792}{1008989} y_{n+1} \\
g_n = {} & \frac{h^2}{15134835}\left\{8484465600 g_{n+\frac{3}{2}} - 1475741025 g_{n+2} + 482520 g_{n+3}\right\} + \frac{h}{15134835}\Big\{-3405691872 f_{n+\frac{1}{2}} + \\
& 21649400 f_{n+\frac{3}{2}} + 77363424 f_{n+\frac{5}{2}} - 318687149 f_n - 230578300 f_{n+1} - 223547175 f_{n+2} - 4383428 f_{n+3}\Big\} \\
& - \frac{7492308075}{1008989} y_{n+2} - \frac{963106560}{1008989} y_{n+\frac{1}{2}} + \frac{42194167040}{3026967} y_{n+\frac{3}{2}} + \frac{373168415}{3026967} y_n - \frac{5484918240}{1008989} y_{n+1}
\end{aligned}\right\} \qquad (9)$$

Equations (7) and (9) is combined to form implicit modified extended hybrid block method for $k = 2$ given as (10)

$$\left.\begin{aligned}
&y_{n+3} = \frac{h^2}{1008989}\left\{9420000g_{n+\frac{3}{2}} - 3138750g_{n+2} - 9150g_{n+3}\right\} + \frac{h}{1008989}\left\{-730080f_{n+\frac{1}{2}} + \right.\\
&\left.27200000f_{\frac{3}{2}} + 972000f_{n+\frac{5}{2}} - 10750f_n - 12453750f_{n+1} + 39555000f_{n+2} + 182240f_{n+3}\right\}\\
&-\frac{179803125}{10008989}y_{n+2} - \frac{4859136}{10008989}y_{n+\frac{1}{2}} + \frac{249120000}{10008989}y_{n+\frac{3}{2}} - \frac{116875}{10008989}y_n - \frac{63331875}{10008989}y_{n+1}\\
&y_{n+\frac{5}{2}} = \frac{h^2}{871766496}\left\{-699684000g_{n+\frac{3}{2}} + 370352250g_{n+2} + 103200g_{n+3}\right\} +\\
&\frac{h}{871766496}\left\{43962480f_{n+\frac{1}{2}} - 2688656000f_{n+\frac{3}{2}} + 96221520f_{n+\frac{5}{2}}\right.\\
&\left.-1062980f_{n+3} - 617470f_n + 810364500f_{n+1} - 3349039500f_{n+2}\right\}\\
&+\frac{623023125}{32287648}y_{n+2} + \frac{344625}{1008989}y_{n+\frac{1}{2}} - \frac{641921500}{27242703}y_{n+\frac{3}{2}} + \frac{6760121}{871766496}y_n + \frac{9923875}{2017978}y_{n+1}\\
&h^2g_{n+\frac{1}{2}} = \frac{h^2}{8717664960}\left\{130423092000g_{n+\frac{3}{2}} - 20200907250g_{n+2} + 5881440g_{n+3}\right\} +\\
&\frac{h}{8717664960}\left\{-108837375408f_{n+\frac{1}{2}} - 532160000f_{n+\frac{3}{2}} + 9829600f_{n+\frac{5}{2}}\right.\\
&\left.-538292f_{n+3} - 619746650f_n - 472315090500f_{n+1} - 31001500f_{n+2}\right\}\\
&\frac{353826375}{193725888}y_{n+2} - \frac{181064150}{3026967}y_{n+\frac{1}{2}} - \frac{10372550}{27242703}y_{n+\frac{3}{2}} + \frac{1552906175}{1743532992}y_n - \frac{5617675}{4035956}y_{n+1}\\
&h^2g_{n+\frac{5}{2}} = \frac{h^2}{26152994880}\left\{173606435200g_{n+\frac{3}{2}} - 7231714050g_{n+2} - 399593760g_{n+3}\right\} +\\
&\frac{h}{26152994880}\left\{-120553737f_{n+\frac{1}{2}} + 593684200f_{n+\frac{3}{2}} + 175707432f_{n+\frac{5}{2}}\right.\\
&\left.+42848164f_{n+3} - 17252782f_n - 2151910100f_{n+1} + 811451100f_{n+2}\right\}\\
&\frac{90830475}{64575296}y_{n+2} - \frac{31280990}{1008989}y_{n+\frac{1}{2}} + \frac{15254670}{81728109}y_{n+\frac{3}{2}} - \frac{3767337805}{5230598976}y_n - \frac{17279765}{4035956}y_{n+1}\\
&g_{n+1} = \frac{h^2}{681067575}\left\{1505942400g_{n+\frac{3}{2}} - 175444650g_{n+2} - 41865g_{n+3}\right\} +\\
&\frac{h}{681067575}\left\{201036096f_{n+\frac{1}{2}} - 3617868800f_{n+\frac{3}{2}} + 7462080f_{n+\frac{5}{2}} + \right.\\
&\left.1785970f_n - 6225216075f_{n+1} + 2826137700f_{n+2} - 387221f_{n+3}\right\}\\
&-\frac{21963161}{1008989}y_{n+2} + \frac{2455040}{1008989}y_{n+\frac{1}{2}} + \frac{1798508032}{27242703}y_{n+\frac{3}{2}} + \frac{823619}{27242703}y_n - \frac{47133792}{1008989}y_{n+1}\\
&g_n = \frac{h^2}{15134835}\left\{8484465600g_{n+\frac{3}{2}} - 1475741025g_{n+2} + 482520g_{n+3}\right\} + \frac{h}{15134835}\\
&\left\{-3405691872f_{n+\frac{1}{2}} + 21649400f_{n+\frac{3}{2}} + 77363424f_{n+\frac{5}{2}} - 318687149\right.\\
&\left.f_n - 230578300f_{n+1} - 223547175f_{n+2} - 4383428f_{n+3}\right\} - \frac{7492308075}{1008989}y_{n+2}\\
&-\frac{963106560}{1008989}y_{n+\frac{1}{2}} + \frac{42194167040}{3026967}y_{n+\frac{3}{2}} + \frac{373168415}{3026967}y_n - \frac{5484918240}{1008989}y_{n+1}
\end{aligned}\right\} \quad (10)$$

Analysis of the IMEHBBDF

The previously mentioned IMEHBBDF was analyzed to confirm its principal numerical characteristics and level of precision, degree of accuracy of the scheme, encompassing the absolute stability region, consistency, zero stability, convergence, linear stability, and local truncation error (LTE). In block form, the IMEHBBDF can be expressed through the following matrix finite difference equation;

$$P^{(1)}Y_{\omega+1} = P^{(0)}Y_{\omega} + h\left(R^{(1)}f_{\omega+1} + R^{(0)}f_{\omega}\right) + h^2\left(Q^{(1)}g_{\omega+1}\right) \quad (11)$$

Where

$$\left.\begin{aligned}
Y_{w+1} &= \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}\right)^T, Y_w = \left(y_{n-\frac{1}{2}}, y_{n-1}, y_{n-\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_n\right)^T\\
F_{w+1} &= \left(f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}\right)^T, F_w = \left(f_{n-\frac{1}{2}}, f_{n-1}, f_{n-\frac{3}{2}}, f_{n-2}, f_{n+\frac{5}{2}}, f_n\right)^T\\
G_{w+1} &= \left(g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{1}{2}}, g_{n+2}, g_{n+\frac{5}{2}}, g_{n+2}\right)^T, G_w = \left(g_{n-\frac{1}{2}}, g_{n-1}, g_{n-\frac{3}{2}}, g_{n-2}, g_{n+\frac{5}{2}}, g_n\right)^T
\end{aligned}\right\}$$

$\omega = 0, \dfrac{1}{2}, 1, \ldots$ *and* $n = 0, 3, \ldots, N-3$

And the matrices $P^{(1)}, P^{(0)}, R^{(1)}, R^{(0)}, Q^{(1)}$ are 6 by 6 matrices whose entries in the coefficients of equation (11) and are respectively defined as follows:

$$\begin{bmatrix} \frac{963106560}{1008989} & \frac{5484918240}{1008989} & -\frac{42194167040}{3026967} & \frac{7492308075}{1008989} & 0 & 0 \\ -\frac{2455040}{1008989} & \frac{47133792}{1008989} & -\frac{1798508032}{27242703} & \frac{21963161}{1008989} & 0 & 0 \\ \frac{31280990}{1008989} & \frac{1727919765}{4035956} & -\frac{152549964670}{81728109} & \frac{90838095475}{64575296} & 0 & 0 \\ \frac{181064150}{3026967} & \frac{561735675}{4035956} & -\frac{10372692550}{27242703} & \frac{35382496375}{193725888} & 0 & 0 \\ -\frac{344625}{1008989} & -\frac{9923875}{2017978} & \frac{641921500}{27242703} & -\frac{623023125}{32287648} & 1 & 0 \\ \frac{4859136}{1008989} & \frac{63331875}{1008989} & -\frac{249120000}{1008989} & \frac{179803125}{1008989} & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{373168415}{3026967} \\ 0 & 0 & 0 & 0 & 0 & -\frac{823619}{27242703} \\ 0 & 0 & 0 & 0 & 0 & \frac{3767337805}{5230598976} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1552906175}{1743532992} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6760121}{871766496} \\ 0 & 0 & 0 & 0 & 0 & \frac{116875}{1008989} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{318687149}{15134835} \\ 0 & 0 & 0 & 0 & 0 & -\frac{357194}{136213515} \\ 0 & 0 & 0 & 0 & 0 & \frac{862606391}{13076497440} \\ 0 & 0 & 0 & 0 & 0 & -\frac{61974665}{871766496} \\ 0 & 0 & 0 & 0 & 0 & -\frac{308735}{435883248} \\ 0 & 0 & 0 & 0 & 0 & \frac{10750}{1008989} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1135230624}{5044945} & \frac{1537187220}{1008989} & -\frac{432980480}{3026967} & -\frac{1490291145}{1008989} & -\frac{25787808}{5044945} & \frac{4383428}{15134835} \\ -\frac{22337344}{75674175} & \frac{27667627}{3026967} & \frac{144714752}{27242703} & -\frac{12560612}{3026967} & -\frac{165824}{15134835} & \frac{387221}{681067575} \\ \frac{93019859}{20179780} & \frac{3985096315}{48431472} & -\frac{18552647290}{81728109} & -\frac{5009004655}{16143824} & -\frac{406730201}{60539340} & -\frac{1071210041}{6538248720} \\ \frac{251938369}{20179780} & \frac{874657575}{16143824} & \frac{166286750}{27242703} & -\frac{578572225}{16143824} & -\frac{455085}{4035956} & \frac{13455523}{2179416240} \\ -\frac{101765}{2017978} & -\frac{7503375}{8071912} & \frac{84020500}{27242703} & \frac{31009625}{8071912} & -\frac{222735}{2017978} & \frac{265745}{217941624} \\ \frac{730080}{1008989} & \frac{12453750}{1008989} & -\frac{27200000}{1008989} & -\frac{39555000}{1008989} & -\frac{972000}{1008989} & -\frac{182240}{1008989} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -\frac{565631040}{1008989} & \frac{98382735}{1008989} & 0 & -\frac{32168}{1008989} \\ 0 & 1 & -\frac{20079232}{9080901} & \frac{259918}{1008989} & 0 & -\frac{2791}{45404505} \\ 0 & 0 & -\frac{3616801115}{54485406} & \frac{892804305}{32287648} & 1 & \frac{832487}{54485406} \\ 1 & 0 & -\frac{271714775}{18161802} & \frac{74818175}{32287648} & 0 & -\frac{12253}{18161802} \\ 0 & 0 & \frac{7288375}{9080901} & -\frac{6858375}{16143824} & 0 & -\frac{1075}{9080901} \\ 0 & 0 & -\frac{9240000}{1008989} & \frac{3138750}{1008989} & 0 & \frac{9150}{1008989} \end{bmatrix}$$

Order and Error Constants of IMEHBBDF

As noted by Fatunla (1991a) and Lambert (1991b), the local truncation error for each method within the IMEHBBDF can be defined using a linear difference operator.

$$\mathcal{L}\{y(x_n); h\} = \sum_{j=0}^{4}\left(\alpha_{\frac{j}{2}}. y_{n+\frac{1}{2}}\right) - \left[h\sum_{j=0}^{6}\left(\beta_{\frac{j}{2}}. f_{n+\frac{j}{2}}\right) + h^2\left\{\gamma_{\frac{\mu}{2}}\, g_{n+\frac{\mu}{2}} + \gamma_k\, g_{n+k} + \gamma_{k+1}\, g_{n+k+1}\right\}\right] \quad (12)$$

If y(t) is sufficiently differentiable, the expression in equation (12) can be obtained by writing the terms as a Taylor series about the point t.

$$L\{y(x); h\} = c_0 y(x_n) + c_1 h y'(x_n) + h^2 c_2 y''(x_n) + \cdots + c_m h^m y^m(x_n) + \cdots \quad (13)$$

Where the constant $c_m,\ m = 0,1,2,$ ...are given as follows:

$$c_0 = \sum_{j=0}^{4} \alpha_{\frac{j}{2}}$$

$$c_1 = \sum_{j=1}^{4}\left(\frac{j}{2}\alpha_{\frac{j}{2}}\right) - \sum_{j=1}^{6}\left(\beta_{\frac{j}{2}}\right)$$

$$c_2 = \sum_{j=1}^{4}\left(\frac{j^2}{2}\ \alpha_{\frac{j}{2}}\right) - \sum_{j=1}^{6}\left(\frac{j}{2}\beta_{\frac{j}{2}}\right) - \gamma_{\frac{1}{2}} - \gamma_2 - \gamma_3$$

$$\vdots$$

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{k+2}\left(\frac{j}{2}\right)^p \alpha_{\frac{j}{2}}\right] - \frac{1}{(p-1)!}\left[\sum_{j=0}^{k+4}\left(\frac{j}{2}\right)^{p-1} \beta_{\frac{j}{2}}\right] + \frac{1}{(p-2)!}\left\{\gamma_{\frac{\mu}{2}}.\left(\frac{\mu}{2}\right)^{P-2} + +\gamma_k.(k)^{P-2} + \gamma_{k+1}.(k+1)^{P-2}\right\}$$

According to Henrici (1962), we say that the methods in (11) have a maximal order of accuracy p if:

$$L\{y(x); h\} = c_{p+1} h^{p+1} y^{p+1}(x_n) + 0(h^{p+2}), c_0 = c_1 = c_2 \ldots c_p = 0, \quad c_{p+1} \neq 0 \quad (14)$$

Therefore c_{p+1} is the error constant and$c_{p+1},\ h^{p+1}, y^{p+1}(x_n)$ the principal local truncation error of each algorithm in each hybrid block mentioned is calculated using (10). Then, the value of the error constant and Order of 2-step implicit modified extended hybrid block BDF-type methods are presented below.

Mathematically, we obtain the order 14 by expanding the main method through Taylor's Series expansion below

$$y(x_n) + 3hy'(x_n) + \frac{9h^2}{2!}y''(x_n) + \cdots - \frac{16875}{1008989}y(x_n) - \frac{4859136}{1008989}\left\{y(x_n) + \frac{h}{2}y'(x_n) + \frac{h^2}{8}y''(x_n) + \cdots\right\} +$$
$$\frac{63331875}{1008989}\left\{y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots\right\} - \frac{249120000}{1008989}\left\{y(x_n) + \frac{3h}{2}y'(x_n) + \frac{9h^2}{8}y''(x_n) + \cdots\right\} +$$
$$\frac{179803125}{1008989}\{y(x_n) + 2hy'(x_n) + 4h^2y''(x_n) + \cdots\} + \frac{10750h}{1008989}y'(x_n) + \frac{730080h}{1008989}\left\{y'(x_n) + \frac{h}{2}y''(x_n) + \frac{h^2}{8}y'''(x_n) +\right.$$

$$\cdots\Big\} + \frac{12453750h}{1008989}\Big\{y'(x_n) + hy''(x_n) + \frac{h^2}{2!}y'''(x_n) + \cdots\Big\} - \frac{27200000h}{1008989}\Big\{y'(x_n) + \frac{3}{2}hy''(x_n) + \frac{9h^2}{8}y'''(x_n) + \cdots\Big\} + \frac{39555000h}{1008989}\{y'(x_n) + 2hy''(x_n) + 2h^2y'''(x_n) + \cdots\} - \frac{972000h}{1008989}\Big\{y'(x_n) + \frac{5}{2}hy''(x_n) + \frac{25h^2}{8}y'''(x_n) + \cdots\Big\} - \frac{182240h}{1008989}\Big\{y'(x_n) + 3hy''(x_n) + \frac{9h^2}{2!}y'''(x_n) + \cdots\Big\} + \frac{3138750h^2}{1008989}\{y''(x_n) + 2hy'''(x_n) + 2h^2y^{iv}(x_n) \ldots\} - \frac{9420000h^2}{1008989}\Big\{y''(x_n) + \frac{3h}{2}y'''(x_n) + \frac{9h^2}{8}y^{iv}(x_n) \ldots\Big\} + \frac{9150h^2}{1008989}\Big\{y''(x_n) + 3hy'''(x_n) + \frac{9}{2!}h^2y^{iv}(x_n) \ldots\Big\} = 0 .$$

By equating the coefficient $y(x_n), y'(x_n), y''(x_n), \ldots, y^p(x_n), y^{p+1}(x_n)$ to zero, we obtain $c_0 = c_1 = c_2 = \cdots = c_p = 0, c_{p+1} \neq 0$ as follows $c_0 = c_1 = c_2 = \cdots = c_{13} = 0,\ c_{14} \neq 0, where\ c_{14} = \frac{15}{28958662340608}$. Conclusively, 2-step IMEHBBDF method is of order nine $\{14, 14, 14, 14, 14, 14\}^T$ with local truncation error $\Big\{\frac{15}{28958662340608}, \frac{325878941}{15012170557371187200}, \frac{-7243}{33360379016380416}, \frac{-139907}{542974918886400}, \frac{-98867}{293206456198656000}, \frac{-4579747}{10008113704914124800}\Big\}^T$ (Lambert, 1973).

Zero Stability

The zero stability of the method is concerned with the stability of the difference system in the limit as h tends to zero see Fatunla (1991). Thus, as h $-\!\!\longrightarrow$ 0 the difference system (11) tends to

$$P^{(1)}Y_{\omega+1} = P^{(0)}Y_{\omega} \tag{15}$$

Any scheme from linear multistep methods (LMM) is zero stable if and only if the determinant of the characteristic equations of the initial matrices equals zero, according to Lambert (1973). Then,

$$\rho(R) = \det\left[RP^{(1)} - P^{(0)}\right] = \frac{484323840000000}{1008989}R^5(1 - R) = 0$$

$$\leftrightarrow\ |R| = 0, 0, 0, 0, 0, 1$$

For$\rho(R) = 0$, the block technique (10) is zero stable, and the multiplicity for those roots with $|R_i| = 1$ does not exceed 1. Therefore, the IMEHBBDF is zero-stable.

Consistency and Convergence of IMEHBBDF

The IMEHBBDF is consistent because its order is greater than1. Convergence, according to Henrici (1962), is consistency plus zero stability. As such, the IMEHBBDF converges.

Linear stability

In the spirit of Hairer and Wanner (1996), the linear stability of the IMEHBBDF is examined and ascertained by applying the test problem.

$$y' = \lambda y, \qquad y'' = \lambda^2 y, \qquad \lambda < 0,$$

On equation (11) to produce

$$Y_{\omega} = W(z)Y_{\omega+1}, \quad z = \lambda h \tag{16}$$

Where the matrix W (z) is given by

$$W(z) = \left(P^{(1)} - zR^{(1)} - z^2Q^{(1)}\right)^{-1}.\left(P^{(0)} + R^{(0)} + Q^{(0)}\right)$$

The matrix W (z) has eigenvalues$\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\} = \{0, 0, 0, 0, 0, \varphi_6\}$, where the dominant eigenvalueφ_6, is the stability function$\varphi(z)\colon \mathbb{C} \to \mathbb{C}$, which is a rational function with real coefficients given by

$$W(z) = \frac{\begin{array}{c}20763859200z^{11}+2332748480z^{10}+4998083820z^{9}-20996349624z^{8}-815344400z^{7}\\ -17083829064z^{6}+133063101825z^{5}-295665540550z^{4}-123807078487z^{3}\\ +4440976236z^{2}+3888340040z-174356582400000\end{array}}{\begin{array}{c}4032572400z^{12}+24195434400z^{11}+297467658120z^{10}+1265546808600z^{9}\\ +4826898637060z^{8}+11980040z^{7}+1165089170z^{6}-192985410z^{5}-771140890z^{4}\\ -13943900150z^{3}+7479511720z^{2}+752456800z-17435682400000\end{array}} \tag{17}$$

The stability domain of the method (or region of absolute stability) S is defined as

$$S = \{z \in\ \mathbb{C}\colon \mathbb{R}(z) \leq 1\} \tag{18}$$

Specifically, when S includes the left half of the complex plane, the method is considered A-stable.

Figure 1 illustrates the regions of (17) in blue and white. The blue stability region clearly demonstrates that the method is A-stable.

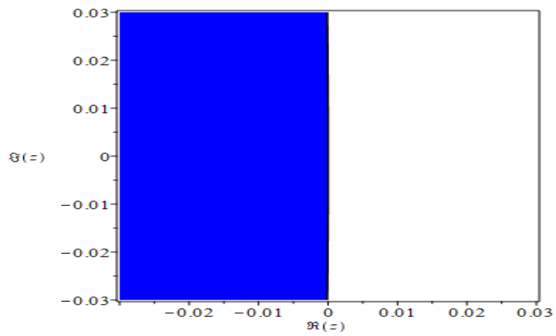


Figure 1: Region of Absolute Stability (RAS)

Test for L-stability of IMEHBBDF

Fatunla, (1973) defines L-stability as

$$\lim_{z \to \infty} W(z) = 0$$

The 3-step IMEHBBDF satisfies the definition above, hence it is L-stable.

Computational Aspect of the EHBBDF

This section presents four problems to demonstrate the validity of the procedure. These four problems are representative DAEs examples with significant implications for applied systems. We determine the maximum absolute error (MAE) and global absolute error (GAE) of the projected solution and the convergence rate of the method. All calculations were performed using our own custom-built Mathematica code within MAPLE 2016.

RESULTS AND DISCUSSION

Numerical Results

The method is implemented more efficiently as an implicit modified extended hybrid block BDF-type numerical integrator for (1) to simultaneously obtain the approximations $\left\{y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}\right\}^T$ without requiring back values or predictors to take $n = 0,\ 6, \dots, N-6$ over sub-intervals$[x_0, x_6], \dots, [x_{N-6}, x_N]$. For example, $n = 0$, $w = 1, \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}\right)^T$ are simultaneously obtained over the sub-interval$[x_0, x_6]$, as y_0 is known from the initial value problem. $n = 1,\ w = 3, \left(y_{n+\frac{7}{2}},\ y_{n+4}, y_{n+\frac{9}{2}},\ y_{n+5,} y_{n+\frac{11}{2}},\ y_{n+6}\right)^T$ Are simultaneously obtained over the sub-interva$[x_6, x_{12}]$, as y_{n+3} is known from the previous block, and so on. The sub-intervals do not overlap as a result. It should be mentioned that the code applies Newton's approach to nonlinear issues.

Numerical Example 1: Consider a Differential Algebraic Equations (DAEs) system of second order Shanmugasundaram P. and Thota S. (2024) and Palanisamy S and Thota S (2023).

$$\left.\begin{array}{r} y_1''(t) = y_1(t) + 2y_2(t) + 2\,t\sin(t) \\ y_2''(t) = -y_2(t) + 2e^t + 2\cos(t) \\ y_3(t) = \cos(t) \end{array}\right\}$$

With initial conditions $y_1(0) = 0, y_2(0) = y_1'(0) = y_2'(0) = 1$. the analytical solution is

$y_1(t) = te^t,\ y_2(t) = e^t + t\sin(t),\ y_3(t) = \cos(t)$

Table 1: Shows Comparison of Global Absolute Error $[y(x_j) - y(x)]$ between Thota and Shanmugasundaram P, 2024 and IMEHBBDF at $\{0 \le t \le 1\}$ *and* $h = 0.1$

t	Thota and Shanmugasundaram, 2024		IMEHBBDF		
	Error 1	Error 2	Error 1	Error 2	Error 3
0.1	3.2×10^{-9}	3.0×10^{-9}	1.085×10^{-19}	3.531×10^{-21}	1.106×10^{-21}
0.2	2.1×10^{-9}	5.0×10^{-9}	1.319×10^{-19}	8.681×10^{-21}	1.703×10^{-21}
0.4	9.5×10^{-9}	2.0×10^{-9}	2.724×10^{-19}	2.326×10^{-20}	3.938×10^{-21}
0.6	3.0×10^{-9}	0	4.662×10^{-19}	7.331×10^{-20}	9.471×10^{-21}
0.8	$1.\ 0 \times 10^{-9}$	0	6.725×10^{-19}	1.282×10^{-19}	1.544×10^{-20}
1.0	$1.\ 0 \times 10^{-9}$	$2.\ 0 \times 10^{-9}$	9.464×10^{-19}	2.137×10^{-19}	2.439×10^{-20}

The developed methods demonstrate superior accuracy and efficiency relative to the approach by Thota and Shanmugasundaram, exhibiting reduced method absolute errors and results that are almost indistinguishable from those of ADM as shown in Table 1 and figures 2a and 2c.

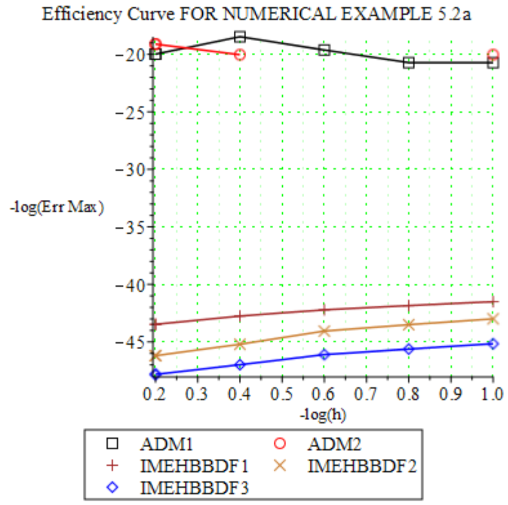
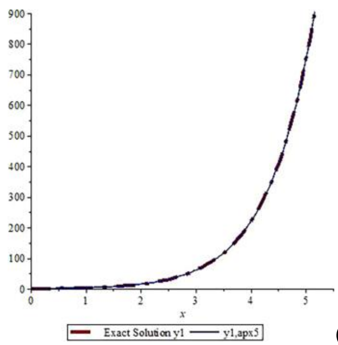
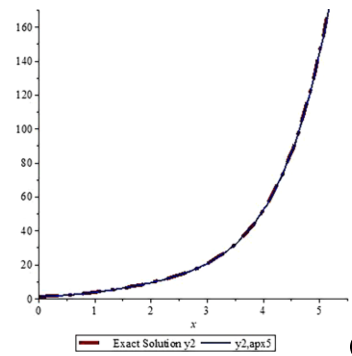


Figure 2 (a): Shows Comparison of Absolute Errors in Term of Efficiency Curve between ADM and IMEHBBDF



(b)



(c)

Figure 2 (b-c): Shows Comparison of Graphs between Exact and Numerical Solutions of ADM (Thota, 2024)

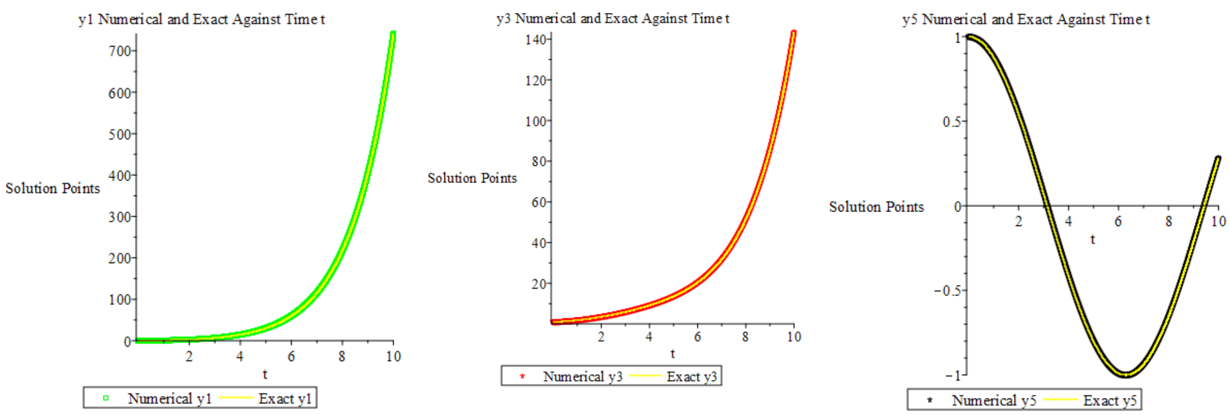


Figure 2 (d): Shows the Perfect Relationship between Exact and Numerical Solutions $y_1(t), y_2(t), y_3(t)$ of IMEHBBDF

In conclusion, Table 1 clearly demonstrates that IMEHBBDF achieved greater accuracy than the Adomian Decomposition Method, as its absolute errors were substantially lower than those reported by Thota, (2024). Additionally, IMEHBBDF generates three numerical graphs for differential states and equality constraint states, whereas the ADM approach in Thota (2024) provides graphs only for differential states.

***Numerical Example 2**:* To demonstrate the suggested method, let's look at the following system of second-order DAEs of index-2 (Thota and Shanmugasundaram, 2024),

$$\left.\begin{array}{c} y_1''(x) - xy_2'(x) = 2y_1(x) - y_2(x) \\ y_2(x) = e^x \end{array}\right\}$$

And initial conditions are $y_1(0) = y_2(0) = y_2'(0) = 1, y_1'(0) = 1$. The exact solution of this system is

$$\begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} = \left.\begin{array}{c} y_1(x) = (2+\sqrt{2})e^{\sqrt{2x}} + (2-\sqrt{2})e^{-\sqrt{2x}} - (x+3)e^x \\ y_2(x) = e^x \end{array}\right\}$$

Table 2: Shows Comparison of Global Absolute Error $|y(x_j) - y(x)|$ between Thota and Shanmugasundaram P, 2024 and IMEHBBDF at $\{0 \le t \le 4\}$ *and* $h = 0.1$

t	Thota and Shanmugasundaram, (2024)	IMEHBBDF	
	Err 1	Err 1	Err 2
0.5	$5{:}778 \times 10^{-06}$	2.6700×10^{-19}	4.0000×10^{-22}
1.0	$4{:}641 \times 10^{-05}$	1.4108×10^{-17}	8.3391×10^{-20}
1.5	$1{:}909 \times 10^{-05}$	2.9279×10^{-17}	1.5215×10^{-19}
2.0	$4{:}643 \times 10^{-05}$	5.8515×10^{-17}	2.7107×10^{-19}
2.5	$3{:}398 \times 10^{-05}$	1.0907×10^{-16}	4.5003×10^{-19}
3.0	$6{:}950 \times 10^{-05}$	1.8212×10^{-16}	6.7360×10^{-19}
3.5	$6{:}850\times 10^{-05}$	2.9730×10^{-16}	9.8744×10^{-19}
4.0	$4{:}870 \times 10^{-05}$	4.9319×10^{-19}	1.4737×10^{-18}

The IMEHBBDF method demonstrates superior accuracy and consistency relative to ADM, as evidenced by Table 2 and Figure 4.24a, which indicate substantially lower absolute errors for the new approach. Likewise, when compared to both the current scheme and ADM, the newly developed curve exhibits the highest level of efficiency among the evaluated methods.

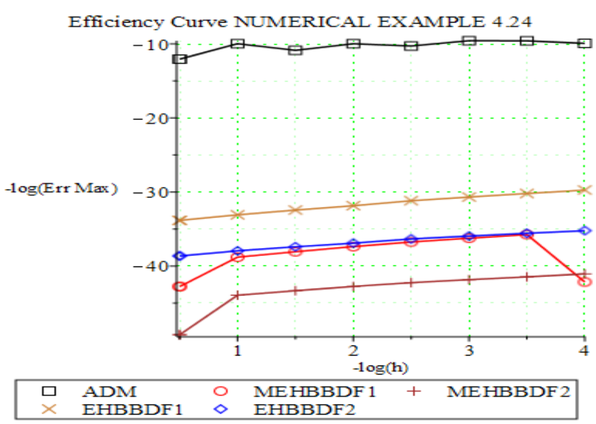


Figure 3(a): Comparison of Efficiency Curves between IMEHBBDF and ADM Methods

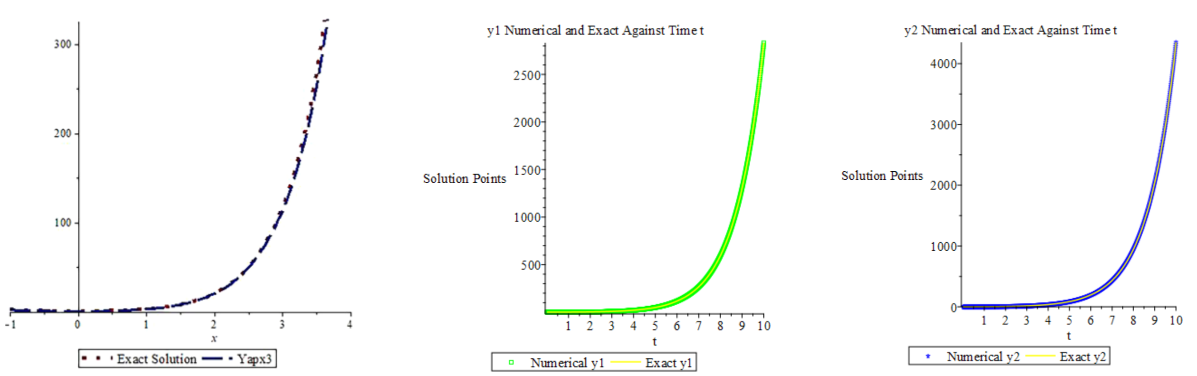


Figure 3(b): Shows Comparison between Exact and Numerical Solutions between ADM and IMEHBBDF

Numerical Example 3: The position of a particle on a circular track is described by an index-three system of differential algebraic equations that we examined. Refer to Sand (2002) and Chin-Shan Liu (2013).

$y_1''(t) = 2y_2(t) + \lambda(t)y_1(t), \qquad y_1(0) = 0$

$y_2''(t) = -2y_1(t) + \lambda(t)y_2(t), \qquad y_2(0) = 1$

$y_1^2(t) + y_2^2(t) - 1 = 0, \qquad \lambda(0) = 0$

With exact solution

$y_1(t) = \sin(t^2), \; y_2(t) = \cos(t^2), \qquad \lambda(t) = -4t^2$

The problem may be regarded as one of mechanical control, involving the choice of a suitable controller $\lambda(t)$ to adjust the system's stiffness, thereby enabling the mechanical system's trajectory to converge over time to a circular path (Chin-Shan Liu, 2013).

Table 3: A Comparison Global Absolute Errors of $y_1(t), y_1(t), \lambda(t)$ between ILGM (2013) and IMEHBBDF

T	ILGM (2013)			IMEHBBDF		
	$\mathbf{Erry_1}$	$\mathbf{Erry_2}$	**Err** λ	$\mathbf{Erry_1}$	$\mathbf{Erry_2}$	**Err** λ
0.0	1.00×10^{-19}	1.00×10^{-13}	1.00×10^{-6}	5.600×10^{-37}	4.400×10^{-29}	2.500×10^{-36}
0.2	1.00×10^{-12}	1.00×10^{-10}	1.00×10^{-5}	2.110×10^{-28}	2.195×10^{-26}	1.419×10^{-27}
0.4	1.00×10^{-11}	$1.00. \times 10^{-10}$	1.00×10^{-5}	1.734×10^{-27}	4.417×10^{-26}	1.165×10^{-26}
0.6	1.00×10^{-11}	1.00×10^{-10}	$1.0l0 \times 10^{-5}$	5.979×10^{-27}	6.702×10^{-26}	4.008×10^{-26}
0.8	1.00×10^{-11}	1.00×10^{-9}	1.00×10^{-5}	1.404×10^{-26}	8.738×10^{-26}	9.549×10^{-26}
1.0	1.00×10^{-11}	1.00×10^{-9}	1.00×10^{-5}	2.707×10^{-26}	1.055×10^{-25}	5.482×10^{-26}

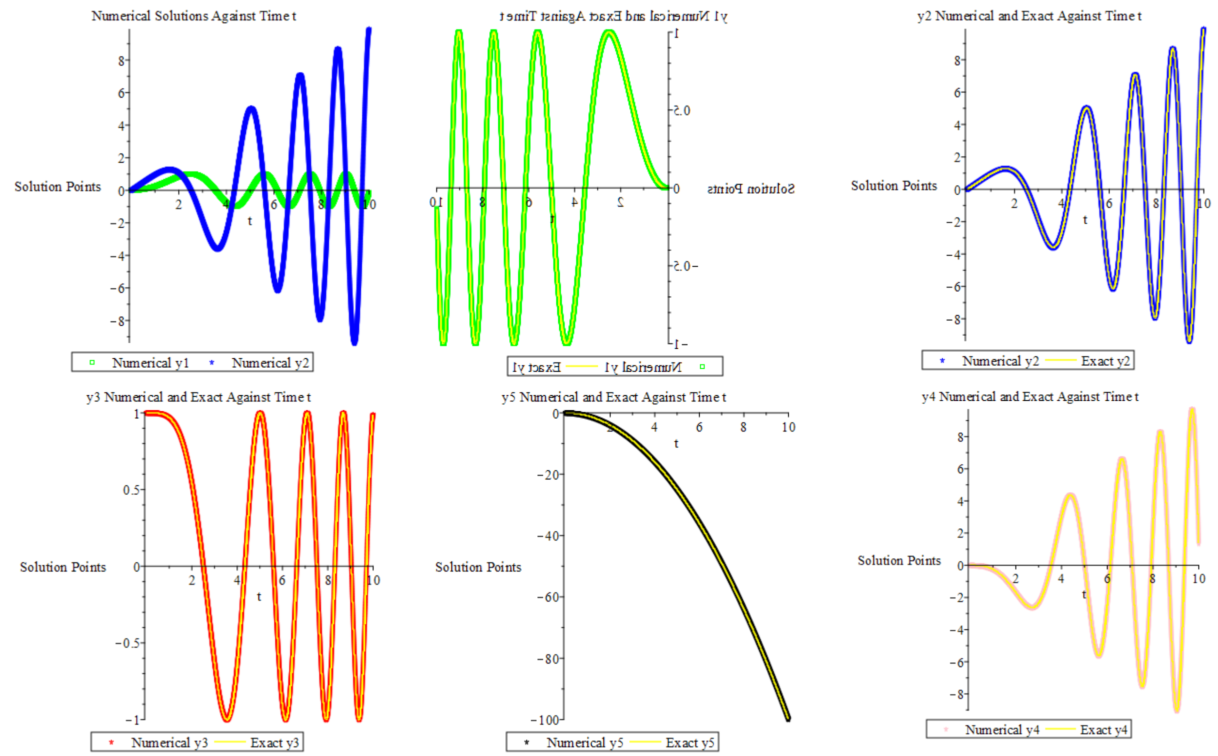


Figure 4: Shows Relationship between Exact and Approximate Solutions with A Wider Range in IMEHBBDF

Frequency and Amplitude, the graph illustrates uniform oscillations that maintain a steady frequency and amplitude. The duration of these oscillations seems to remain constant, with each full cycle requiring approximately 2 times be steady, with each complete cycle taking about 2-time units. Phase Relationship, the numerical and exact solutions are not just nearly equal in value but also exhibit an insignificant phase difference. This indicate that the timing of the oscillations is almost the same for both solutions.

Based on Figure 4, regarding the relationship of phases, both the numerical and exact solutions are not only close show similar value but also exhibit no significant phase discrepancy. This indicates that the oscillations timing for the two solutions is almost the same. Additionally, concerning frequency and amplitude, the graph illustrates uniform oscillations that maintain with a steady frequency and amplitude. The duration of these oscillations seems constant, with each complete cycle lasting approximately 2-time units. In summary, the

graph successfully indicates that the numerical method utilized offers a highly accurate estimation of the precise approximate solution for $y_1(t), y_2(t), y_3(t), y_4(t), y_5(t)$ within the specified time. The modeled system is characterized as a stable, oscillatory system that shows no significant damping or variation in amplitude over time. This close correspondence signifies the reliability of the numerical technique used.
We analyzed a set of differential algebraic equation with an index of three, which represent the position of a particle traveling along a circular trajectory as described by Sand (2002) and Chin-Shan Liu (2013). This specific problem can be seen as a mechanical control problem, where the innovative methods shown in figure 4 are applied to choose an appropriate controller by $\lambda(t)$ to change the stiffness of the system, enabling the mechanical system's path to accurately trace a circle over time. We implemented the previously discussed numerical method. In solving Numerical Example 3, Table 3 displays the absolute errors, demonstrating that the numerical outcomes show a much higher degree of precision and effectiveness compared to the technique proposed by Chin-Shan Liu (2013). The result from Numerical Example 3 emphasizes the success of the new methodology, which can be conveniently utilized to address nonlinear differential-algebraic equations remarkable precision and effectiveness.

Numerical Example 4: The fourth problem is an index-1 Hessenberg DAE system with nonlinear differential equations and a nonlinear algebraic equation, as follows see Ampon Dhamacharoen, (2016)

$$\left.\begin{aligned} y_1''(t) &= -(3t+1)\,y_2(t) - y_1(t)(4\,y_3(t)+1), 0 \le t \le 1 \\ y_2''(t) &= 4cos(y_3(t)) - y_2(t)(4\,y_3(t)+1) \end{aligned}\right\}$$

Initial condition

$y_1(0) = 1, y_1'(t) = 1, y_2(0) = 0, y_2'(0) = 1$

Algebraic equation

$4\,y_1(t)cos(y_3(t)) + t\,y_2^2(t) = 4(y_3(t) - t^2)$

Using MAPLE, 2016, this problem is solved and has a nice result with a small error as compared to the exact solution

$y_3(t) = t^2 + t,\ y_1(t) = t\cos(t^2+t),\ y_2(t) = sin(t^2+t)$

Table 4: Comparison of Global Absolute Errors $y_1(t), y_2(t), y_3(t)$ between Ampon Dhamacharoen (2016) and IMEHBBDF

t	The Newton-Broyden Method, (2016)			IMEHBBDF		
0.00	0.000000	0.000000	0.000000	6.070×10^{-31}	2.860×10^{-28}	6.080×10^{-31}
0.08	0.082994	0.180310	0.090278	2.770×10^{-31}	7.790×10^{-31}	2.800×10^{-31}
0.17	0.163526	0.386443	0.194444	1.450×10^{-27}	1.770×10^{-26}	1.490×10^{-31}
0.25	0.237892	0.614877	0.312500	3.320×10^{-27}	2.680×10^{-26}	3.500×10^{-27}
0.33	0.300950	0.859913	0.444444	5.720×10^{-27}	3.400×10^{-26}	6.140×10^{-27}
0.42	0.346161	1.113184	0.590278	9.220×10^{-27}	4.130×10^{-26}	1.020×10^{-27}
0.50	0.365844	1.363278	0.750000	1.290×10^{-26}	4.580×10^{-26}	1.460×10^{-26}
0.58	0.351717	1.595568	0.923611	1.700×10^{-26}	4.820×10^{-26}	1.990×10^{-26}
0.67	0.295777	1.792384	1.111111	2.230×10^{-26}	4.830×10^{-26}	2.760×10^{-26}
0.75	0.191575	1.933653	1.312500	2.680×10^{-26}	4.370×10^{-26}	3.480×10^{-26}
0.83	0.035838	1.998150	1.527778	3.130×10^{-26}	3.510×10^{-26}	4.320×10^{-26}
0.92	−0.169652	1.965449	1.756945	3.620×10^{-26}	2.010×10^{-26}	5.400×10^{-26}
1.00	−0.416147	1.8185950	2.0000002	3.980×10^{-26}	7.100×10^{-28}	6.460×10^{-26}

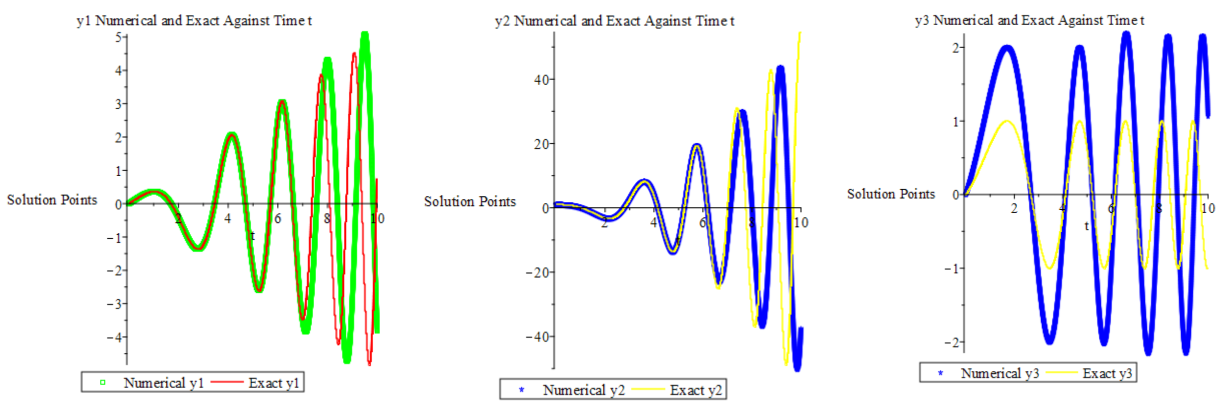


Figure 5: Shows Comparison between Exact and Approximate Solutions of IMEHBBDF with a Wider Range

As time t increases, the numerical solution diverges significantly from the exact solution. The amplitude of the numerical solution grows larger, while the exact solution maintains a consistent amplitude. This divergence suggests that the numerical method may be unstable or inaccurate over longer periods, possibly due to cumulative errors or a mismatch in capturing the system's dynamics. Regarding frequency and amplitude, the frequency of oscillation remains fairly consistent for both the numerical and exact solutions, but the amplitude differs significantly, especially after$t = 4$. With respect to be a phase shift, a noticeable shift develops between the numerical and exact solutions, particularly after$t = 6$. This phase difference further highlights the deviation between the two solutions.

From figure 5, the graphs highlight a growing discrepancy between the numerical and exact solutions for $y_1(t), y_2(t), y_3(t)$ over time. While both solutions start similarly, the numerical solution (blue curve) deviates significantly as time progresses, with increasing amplitude and a phase shift relative to the exact solution (yellow curve). This suggests that the numerical method used may not be stable or suitable for long-term predictions in this context, leading to errors that accumulate over time.

CONCLUSION

With the intention of solving nonlinear second-order DAE systems by converting DAEs into ODE systems via differentiation-index, the IMEHBBDF method was developed. The methods coupled the main and complementary methods to solve the differential algebraic systems connected with the problem. The numerical method was found to be more accurate and efficient than the existing method, demonstrating its potential for solving nonlinear second-order differential algebraic equations.

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