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Determination of the Meson Spectra in the Dirac Equation with Power Law Potential

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ABSTRACT

The pseudo-shifted ℓ-expansion technique was used in the Dirac equation to derive a Schrodinger-like equation. The equation was solved using a non-QCD based power potential of the form; $V(r) = (g_1 r^a - V_o)$ with $a = \frac{m_o}{2m_q}$ to obtain the mass spectra for bottom quark γ, charmonium Ψ, up quark ρ, $s$trange quark ∅ and mixed quark φ. The mass spectra of both light and heavy mesons obtained in this work were in good agreement with both the experiment and other theoretical works. However, the mass spectra of 10.1564 and 10.3039 obtained for bottom quark were small when compare with the experimental results for the same orbits. Other discrepancies observed were the 9.9252 for bottom quark and 3.9715 for charmonium that were higher than experimental results. This model has successfully mimicked the mass spectra of both light and heavy mesons and also predicts the mass spectra of mixed mesons.

Keywords:
Spectra,
Mesons,
Dirac equation,
PSLET,
Power potential.

INTRODUCTION

The power law potential models have been used previously in the Schrodinger equation to predict the mass spectroscopy, density of mesons successfully (Kang and Schnitzer, 1975; Ram and Halasa, 1979). The authors used linear and harmonic oscillator potentials to calculate the ground state energies of $\varphi, \Phi \; and \; \rho$ mesons. Further, the spectra of Υ and φ were explained by the use of the fractional potential (Jena, 1983; Jena and Tripati, 1983)

The Schrodinger equation for some specific cases has a solution or an approximation, and solving it yields the Eigenvalues and Eigen functions for the energy and the spectroscopy of mesons and other physical quantities. For some particular cases, the Schrodinger equation has an exact solution, but in some special cases, numerical techniques or approximation schemes are employed; methods like the WKB methods, Nikoforov Uvarov method, and PSLET (pseudo-perturbative shifted-ℓexpansion technique) method amongst numerous methods used (Sharma and Sharma, 1984; Mustafa, 2003).

These numerical methods have been used successfully to analyze the mass spectroscopy of the meson employing the Nikoforov-Uvarov method and WKB successfully (Hall, 2005; Sharma and Fiase, 2003; Ikhdair and Falaye, 2014). The PSLET method has been implemented to analyze the meson spectroscopy (Mustafa and Znojil, 2002). Some of these methods are not widely applicable, that is to say, some of the methods cannot approximate efficiently for some problems. Although some methods give simple relations for the eigenvalues but give very complicate relations for the Eigen function.

The simple way for finding both eigenvalues and the Eigen functions of Schrodinger and Schrodinger-like equations for power-law and logarithmic potentials, which are very important in particle physics has been investigated (William, Inyan and Thompson, 2020; Ushie, 2021). Numerous works have been carried out using different approximation methods. In this work, the researchers seek to test the validity of the PSLET recipe using the non-quantum chromodynamics (QCD) power law potential and predict the mass spectroscopy of heavy, light and mixed meson.

MATERIALS AND METHODS

The Schrödinger equation is solved numerically different methods ranging from harmonic method, PSLET, variation method, Wentzel-Krammers-Brillioun approximation, among many others. The research wishes to use the PSLET due to its robustness and

newness as it provides a reliable result as tested by others that have worked on this research (Mustafa and Odeh, 2001; Mustafa, 2004).
The power law potential of the form was employed in this work. (Sharma and Fiase, 2003)

$$V(r) = (g_1 r^a - V_o) \quad (1)$$

Where $a = \frac{m_o}{2m_q}$,

m_q is the mass of the constituent quarks (in GeV) and m_o the parameter that render dimensionless setting m_o=1GeV
The above potential is an admixture of scalar and vector parts as suggested by the phenomenology of fine, hyperfine splitting of heavy quarkonium system in non-relativistic approach (Magyari, 1980).

Dirac Bound State [PSLET Recipe]

The Dirac equation with Lorentz Scalar [added to the mass term] and Lorentz vector [coupled as the O-component of the 4-vector potential] potential reads in (ħ = c = 1units) has been used.

$$\{\vec{\alpha}.\vec{\beta} + \beta[m + S(r)]\}\varphi(\vec{r}) = \{E - V(r)\}\varphi(r) \quad (2)$$

Since $V_r(r)$ and $V_s(r)$ are spherical symmetry, equation (2) can be separated in a system of the following coupled equations for the radical wave functions (Schiff, 1968). That is $\emptyset(r) and\ \gamma(r)$ as shown in equations (3) and (4):

$$\left[E' - V_r - V_s - m_q\right]\emptyset(r) + \left\{\frac{k+1}{r} + \frac{d}{dr}\right\}\gamma(r) = 0 \quad (3)$$

$$\left[E' - V_r + V_s + m_q\right]\gamma(r) + \left\{\frac{k-1}{r} - \frac{d}{dr}\right\}\emptyset(r) = 0 \quad (4)$$

Equation (3) can also be written in the form:

$$\varepsilon_{1(r)} G_{(r)} + \frac{dF(r)}{dr} - \frac{k}{r} F(r) = 0 \quad (5)$$

$$\varepsilon_{2(r)} F_{(r)} + \frac{dG(r)}{dr} - \frac{k}{r} G(r) = 0 \quad (6)$$

where:

$$\varepsilon_1 = \left[E' - V_r - V_s - m_q\right]\emptyset(r) = G(r) \quad (7)$$

$$\varepsilon_2 = \left[E' - V_r + V_s + m_q\right]\gamma(r) = F(r) \quad (8)$$

The key points about the power law potential $V_{(r)} = g_o r^{(\frac{m_o}{2mq})} - V_o$, using the Dirac equation for an independent quark model, Magyari (1980) suggested the potential be treated like this:

$$V'(r) = \frac{1}{2}V(r) = V_s(r) + V_r(r) \quad (9)$$

With the choice of the vector fraction gv = ½, the potential for the independent particle model of quarks would approximately read as:

$$V'(r) = \frac{1}{2}V(r) = V_s(r) + V_r(r) \quad (10)$$

$$V_s(r) = S(r)\ ; V_r(r) = V(r) \quad (11)$$

For this case, each of the scalar and vector parts would be equal to 1/4V(r)
Now, where k = $-(l + 1)$ for j = l + ½,
$k = l \; for \; j = l$ – ½
Thus

$$\varepsilon_{1(r)} = E - V(r) - [m + S(r)] \quad (12)$$

$$\varepsilon_{2(r)} = E - V(r) + [m - S(r)] \ \ldots E + m - y(r) \quad (13)$$

$$y(r) = V(r) - S(r) \quad (14)$$

From equation (6), we make F(r) subject. We know that E is the relativistic energy while G(r) and F(r) are the large and small radial components of the Dirac Spinor.

$$F(r) = \frac{\left[\frac{dG(r)}{dr} + \frac{k}{r}\right]}{\varepsilon_{2(r)}}$$

Substitute the above expression in equation (5), we obtained:

$$\varepsilon_{1(r)} G_{(r)} + \frac{d}{dr}\left[\frac{1}{\varepsilon_{2(r)}}.\frac{dG(r)}{dr} + \frac{k}{r}\frac{G(r)}{\varepsilon_{2(r)}}\right] - \frac{k}{r}\left[\frac{1}{\varepsilon_{2(r)}}.\frac{dG(r)}{dr} + \frac{k}{r}\frac{G(r)}{\varepsilon_{2(r)}}\right] = 0 \quad (15)$$

Using product rule for the required differentiation in equation (5)

$$\varepsilon_{1(r)} G_{(r)} + \frac{d}{dr}\left[\frac{1}{\varepsilon_{2(r)}}.\frac{dG(r)}{dr}\right] + \frac{d}{dr}\left[\frac{k}{r}\frac{G(r)}{\varepsilon_{2(r)}}\right] - \frac{k}{r}.\frac{1}{\varepsilon_{2(r)}}\frac{dG(r)}{dr} + \frac{k^2}{r^2}\frac{G(r)}{\varepsilon_{2(r)}} = 0$$

Upon differentiation, we have

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + \frac{y'(r)}{\varepsilon_2(r)}.\frac{d}{dr} + \frac{y'(r)}{\varepsilon_2(r)}.\frac{k}{r} + \varepsilon_1(r)\varepsilon_2(r)\right] G(r) = 0 \qquad (16)$$

Now

$$\frac{d^2G(r)}{dr^2} + \frac{y'(r)}{\varepsilon_2(r)}.\frac{dG(r)}{dr} - \left[\frac{k(k+1)}{r^2} + \frac{k}{r}.\frac{y'(r)}{\varepsilon_2(r)}\right] G(r) = -\varepsilon_1(r)\varepsilon_2(r)G(r)$$

Where $y' = \frac{d}{dr}$, $G(r) = \emptyset exp\left[\frac{-p(r)}{2}\right]$ and $P'(r) = \frac{y'(r)}{\varepsilon_2(r)}$

It has been established in this work that

$$\left[-\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} + u(r) - \varepsilon_1(r)\varepsilon_2(r)\right] \emptyset(r) = 0 \qquad (17)$$

Where

$$u(r) = \frac{y''(r)}{2\varepsilon_2(r)} - \frac{k}{r}\frac{y'(r)}{\varepsilon_2(r)} + \frac{3}{4}\left[\frac{y'(r)}{\varepsilon_2(r)}\right]^2 \qquad (18)$$

This can reduce to Klein Gordon equation with $k(k+1) = l(l+1)$ for any K, if $u(r)$ is set to zero (Mustafa, 2008). It is therefore convenient to introduce a parameter $\Lambda = 0,1$ in $u(r)$ so that $\Lambda = 0$. Also, we shall be interested in the problems where the rest energy mc^2 is large compared to the binding energy (Magyari, 1980)

$E_{bind} = E - mc^2$

Would evolve to

$$\frac{1}{\varepsilon_2(r)} = \frac{1}{E_{bind} + 2m - y(r)} \approx \frac{1}{2m} - 0(\frac{1}{m^2}) \qquad (19)$$

Which turns to:

$$u(r) = \frac{\Lambda}{4m}\left[y''(r) - \frac{2ky'(r)}{r} + \frac{3y'(r)^2}{4m}\right] \qquad (20)$$

Considering the Columb like potentials

$V(r) = -\frac{A_1}{r}$ *and scalar* $S(r) = -\frac{A_2}{r}$ (Lorentz scalar and potential vectors)

$V(r) = V(r)^2 - \frac{A_1^2}{r^2}$ - - - - (a)

$S(r) = S(r)^2 - \frac{A_2^2}{r^2}$ - - - - (b)

Equation (18) further elaborates to

$$-\frac{d^2}{dr^2} + \left[\bar{l}^2 + \bar{l}(2\beta_o + 1) + \beta_o(\beta_o + 1) + \Gamma(r) + 2EV(r)\right]\emptyset(r) = \varepsilon^2\emptyset(r) \text{ -} \qquad (21)$$

Where

$$\Gamma(r) = -V_r(r) + S_r(r) + 2mS(r) + m^2 + u(r) \qquad (22)$$

$$\bar{l} = l' - \beta_o;\ l' = -\frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - A_1^2 + A_2^2} \qquad (23)$$

And β_o is a suitable shifts to be determined, setting:

$x = \bar{l}^{\frac{1}{2}}(r - r_o)/r_o$, where r_o is currently an arbitrary point to be determined through the minimization of the lending energy term below. It is therefore convenient to expand about $x = o$ ($i.e\ r = r_o$) and use the following expansions.

$$\sum_{n=0}^{\infty} \frac{a_n}{r_0^2} x^n \bar{l}^{-\frac{n}{2}};\ a_n = (-1)^n(n+1) \qquad (24)$$

$$\Gamma_x(r) = \frac{\bar{l}^2}{Q}\sum_{n=o}^{\infty} b_n x^n \bar{l}^{-\frac{n}{2}} \qquad (25)$$

$$b_n = \frac{d^n\Gamma(r_o)}{dr_o^n}\frac{r_o^n}{n!} \qquad (26)$$

$$Vx(r) = \frac{\bar{l}}{\sqrt{Q}}\sum_{n=0}^{\infty} c_n x^n \bar{l}^{-\frac{n}{2}};\ c_n = \frac{d^nV(r_o)}{dr_o^n}\frac{r_o^n}{n!} \qquad (27)$$

$$E = \frac{1}{\sqrt{Q}} \sum_{n=-1}^{\infty} E^{n} \bar{l}^{-n} \tag{28}$$

Where Q is set equal to $\bar{l}^2$ at the end of the calculations. With the above expression into (x) one may collect all x-independent terms of order $\bar{l}$ to imply the leading-order approximation for the energies (Mustafa, 2003).

$$\varepsilon^{(-l)} = V(r) \pm \sqrt{V(r_o)^2 + \Gamma(r_o) + \frac{\emptyset}{r_o^2}} \tag{29}$$

Which upon minimization, i.e. $\frac{dE^{(-l)}}{dr_o} = 0$ and $\frac{d^2E^{-l}}{dr_0^2} > 0$

$$2Q = h(r_o) + \sqrt{h(r_o)^2 - g(r_o)} \tag{30}$$

Where

$$h(r_o) = r_o^3\left[2V(r_o)V' + \Gamma'(r_o) + r_oV'(r_o{}^2)\right] \tag{31}$$

$$g(r_o) = r_o^6\left[\Gamma'(r_o)^2 + 4V(r_o)V'(r_o)\Gamma'(r_o) - 4\Gamma(r_o)V'(r_o)^2\right] \tag{32}$$

And primes denote derivatives with respect to r_o, this implies that $x\bar{l}^{-l}$. Coefficients vanish, i.e.

$$Q_{a1} + r_o^2b_1 + 2r_o^2E^{(-1)}c_1 = 0 \tag{33}$$

Equation (21) reduces to

$$\left[-\frac{d^2}{dx^2} + \sum_{n=2}^{\infty}\Gamma_n x^n\bar{l}^{-\frac{n-2}{2}} + (2\beta_o + 1)\sum_{n=0}^{\infty}a_nx^n\bar{l}^{-\frac{n}{2}} + \beta_o(\beta+1)\sum_{n=0}^{\infty}a_nx^n\bar{l}^{-\frac{n}{2}}\right] + \frac{2r_o^2}{Q}\sum_{n=0}^{\infty}\sum_{p=0}^{n+1}E^{(n-p)}\left[c_{2p}x^{2p}\bar{l}^{-n} + c_{2p}x^{2p+1}\bar{l}^{-(n+1)/2}\right]\emptyset_{k,l(x)} = \left[\frac{r_o^2}{Q}\sum_{n=-1}^{\infty}\sum_{p=-1}^{n+1}E^{(n-p)}E^{(p)}\bar{l}^{(n+1)}\right]\emptyset_{k,l(x)} \tag{34}$$

$$\text{Where } T_n = a_n + \frac{r_o^2}{Q}b_n \tag{35}$$

Equation (34) can be compared with Schrodinger's equation for one dimensional harmonic oscillator.

$$\left[-\frac{d^2}{dy^2} + \frac{1}{4}\omega^2y^2 + \varepsilon_o + \beta_{(y)}\right]Y_k(y) = \mu_xY_k(y) \tag{36}$$

Where ε_o is constant, $\beta_{(y)}$ is a perpetuation like term and

$$\mu_x = \varepsilon_o + \left(k + \frac{1}{2}\right)\omega + \sum_{n=1}^{\infty}\mu^{(n)}\bar{l}^{-n} \tag{37}$$

K=0,1,2 … and 2r

$$\omega = \sqrt{12 + \frac{2r_o^4}{Q}\Gamma''(r_o) + \frac{4r_o^4}{Q}E^{(-1)}V''(r_o)} \tag{38}$$

One can further show that

$$E^{(0)} = \frac{Q}{2r_o^2(E^{(-1)} - c_o)}\left[\left(2\beta_o + 1\right) + (k + \tfrac{1}{2})\omega\right] \tag{39}$$

And choose β_o so that $E^{(0)} = 0$ to have

$$\beta_o = -\frac{1}{2}\left[1 + \left(k + \frac{1}{2}\right)\omega\right] \tag{40}$$

Equation (35) becomes

$$\left[-\frac{d^2}{dx^2} + \sum_{n=0}^{\infty}\left(V^{(n)}(x)\bar{l}^{-\frac{n}{2}} + J^{(n)}(x)\bar{l}^{-n} + K^{(n)}(x)\bar{l}^{-\left(n+\frac{1}{2}\right)} + \epsilon^{(n)}\,\bar{l}^{-(n+1)}\right)\right]\emptyset_{k,l}(x) = 0 \tag{41}$$

From equation (41), the following equations were obtained:

$$V^{(0)}(x) = T_2x^2 + (2\beta_o + 1)a_o \tag{42}$$

$$V^{(1)}(x) = T_3x^3 + (2\beta_o + 1)a_1x \tag{43}$$

$$V^{(n)}(x) = T_{n+2}x^{n+2} + \left(2\beta_o + 1\right)a_nx^n + \beta_o\left(\beta_o + 1\right)a_{n-2}x^{n-2} \quad n \geq 2 \tag{44}$$

$$J^{(n)}(x) = \left(\frac{2r_o^2}{Q}\right)\sum_{p=0}^{n+1}E^{(n-p)}\,c_{2p}x^{2p} \tag{45}$$

$$K^{(n)}(x) = \left(\frac{2r_o^2}{Q}\right)\sum_{p=0}^{n+1}E^{(n-p)}\,c_{2p+1}x^{2p+1} \tag{46}$$

$$\epsilon^{(n)} = \left(\frac{r_o^2}{Q}\right)\sum_{p=-1}^{n+1}E^{(n-p)}\,E^p \tag{47}$$

Using closely PSLET recipe for the k-nodal wave function and define

$$\emptyset_{k,l}(x) = F_{k,l}(x)exp(U_{kl}(x)) \tag{48}$$

Where

$$F_{k,l}(x) = \sum_{n=0}^{\infty}\sum_{p=0}^{k=1} A_{p,k}^{(n)}\, x^p \bar{l}^{-n/2} \tag{49}$$

$$U'_{k,l}(x) = \sum_{n=0}^{\infty}\left(U_{k,l}^{(n)}(x)\bar{l}^{-\frac{n}{2}} + G_{k,l}^{(n)}(x)\bar{l}^{-\frac{n+1}{2}}\right) \tag{50}$$

With

$$U_{k,l}^{(n)}(x) = \sum_{p=0}^{n+1}\left(D_{p,n,k}x^{2p-1}\right); \quad D_{0,n,k} = 0 \tag{51}$$

$$G_{k,l}^{(n)}(x) = \sum_{p=0}^{n+1} c_{p,n,k}x^{2p} \tag{52}$$

Equation (31) reads

$$F_{k,l}(x)\sum_{n=0}^{\infty}\left[V^{(n)}(x)\right]\bar{l}^{-\frac{n}{2}} + J^{(n)}(x)\bar{l}^{-n} + K^{(n)}(x)\bar{l}^{-\left(n+\frac{1}{2}\right)} - \in^{(n)}\bar{l}^{-(n+1)}] - F_{k,l}(x)\left[U''_{k,l}(x) + U'_{k,l}(x)U'_{k,l}(x)\right] - 2F'_{k,l}(x)U'_{k,l}(x) - F''_{k,l}(x) = 0 \tag{53}$$

Where the primes denote derivatives with respect to x. one may also eliminate $\bar{l}$- dependence from equation (53) to obtain four exactly solvable recursive relations.

Establishing an Eigen Function from the Power Law Potential

Consider an equally mixed scalar and vector power law potential of the form

$$V(r) = g_o r^a - V_o \tag{54}$$

Where $a = \frac{m_o}{2_{mq}}$

Thus

$$V(r) = \frac{1}{4}(g_o r^a - V_o) \tag{55}$$

But from equation (a) and (b) above we know that:

$$\Gamma(r) = -V_r(r) + S_r(r) + 2mS(r) + m^2 + U(r)$$

$$U(r) = \frac{\lambda}{4m}\left[y'(r) - \frac{2Ky'(r)}{r} + \frac{3y'(r)^2}{4m}\right] \tag{56}$$

For an equally mixed potential; $U(r) = 0; y(r) = V(r) - S(r)$

But since $V(r) = S(r)$

$$y(r) = V(r) - S(r) = 0 \tag{57}$$

The equation (56) yields

$$\Gamma(r) = -V(r)^2 + \frac{A_1^2}{r^2} + S(r)^2 - \frac{A_2^2}{r^2} - 2m\left(\frac{1}{4}(g_o r^a - V_o)\right) + m^2 \tag{58}$$

We have developed that $V(r) = S(r)$, hence

$$\Gamma(r) = \frac{A_1^2}{r^2} - \frac{A_2^2}{r^2} - 2m\left(\frac{1}{4}(g_o r^a - V_o)\right) + m^2 \tag{59}$$

Where $A_1 = A_2\ and\ m = m_o$

$$\Gamma(r) = 2m\left(\frac{1}{4}(g_o r^a - V_o)\right) + m_q^2 \tag{60}$$

$$\Gamma(r) = \left[\frac{m_q g r^a - V_o m_q}{2}\right] + m_q^2 \tag{61}$$

Differentiating the above equation (61) we have

$$\Gamma'(r) = \frac{a m_q g r^{(a-1)}}{2} \tag{62}$$

Differentiating the potential; $V(r) = \frac{1}{4}(g_o r^a - V_o)$ yields

$$V'(r) = \frac{1}{4} a g_1 r^{(a-1)} \tag{63}$$

Recall that

$$2Q = h(r) + \sqrt{h(r)^2} - g(r) \tag{64}$$

But

$$g(r) = r^6\left[\frac{4m^2A^2}{r^4} + 4\left[\left(-\frac{A}{r}\right)\left(\frac{A}{r^2}\right)\left(\frac{2mA}{r^2}\right)\right] - 4\left(\frac{-2mA}{r} + m^2\right)\left(\frac{A^2}{r^4}\right)\right] \tag{65}$$

$$g(r) = r^6\left[\frac{4m^2A^2}{r^4} - \frac{8A^3m}{r^5} + \frac{8A^3m}{r^5} - \frac{4m^2A^2}{r^4}\right] \tag{66}$$

$$g(r) = 0$$

Where $h(r) = r^3[2v(r)V'(r) + \Gamma(r) + rV'(r)^2]$

$$h(r) = r^3\left[2\left(\frac{ag_1^2 r^{(2q-1)}}{16} - \frac{ag_1 V_o r^{(a-1)}}{16}\right) + \frac{am_q g_1 r^{(a+2)}}{16} + \frac{a^2 g_1^2 r^{(2a+2)}}{16}\right] \tag{67}$$

Thus:

$2Q = h(r) + \sqrt{h(r)^2} - 0$

$Q = h(r) = 16Q(2ag_1^2 + a^2g_1^2)r^{(2a+2)} + (8ag_1 - 2ag_1V_o)r^{(a+2)}$ (68)

$Q = (0.5ag_1^2 + 0.065a^2g_1^2)r^{(2a+2)} + (ag_1 - 0.5ag_1V_o)r^{(a+2)}$ (69)

Recall

$E^{(-1)} = \left[\frac{1+r^4\Gamma''(r)}{4}\right]\frac{Q}{r^2}$ (70)

Upon minimization,

$\frac{dE^{(-1)}}{dr} = 0$ (71)

$Q = 0.125(a-1)a^2m_qg_1r^{(a+2)}$ (72)

From the above expressions, we have

$r = \left[\frac{0.125(a-1)a^2m_qg_1-(ag_1-0.5ag_1V_o}{0.5ag_1^2+0.0625a^2g_1^2}\right]^{\frac{1}{a}}$ (73)

Recall that

$\frac{\omega^2}{4} = 3 + \frac{r^4}{2Q}\Gamma''(r)$ (74)

Recall that

$Q = l - \left(\frac{1}{2}\right)\left[1 + (k + \frac{1}{2})\omega\right]^2$ (75)

Hence Q becomes

$Q = \left[l + \frac{1}{2}\left(1 + \left(n - l - \frac{1}{2}\right)\left(\frac{12a+8}{a}\right)^2\right)\right]^2$ (76)

$E = \frac{1}{\sqrt{Q}}\sum_{n=1}^{\infty} E^{(n)}\bar{l}^n$ (77)

$= \frac{1}{\sqrt{Q}}E^{-1}\bar{l}$ (78)

$E = E^{(-1)}$

But

$E^{(-1)} = V(r) \pm \sqrt{V(r)^2} + \Gamma' + \frac{Q}{r^2}$ (79)

And $m = 2E$ (80)

An algorithm was written using equations (61), (73), (79) and (80) in an interactive problem-solving environment of Maple-18. The parameters in Table 1 as well as the respective bound state mass of $\gamma, \varphi, \emptyset\ and\ \rho$ were then obtained and the results are shown in Table 2 to 6.

RESULTS AND DISCUSSION

Results

Table 1: Parameters generated for light, mixed and heavy mesons

Light Meson (GeV)			Mixed meson (GeV)			Heavy meson (GeV)		
g_1	v_0	r	g_1	v_0	r	g_1	v_0	r
0.026	0.745	5.038	0.264	1.035	8.765	0.065	0.167	15.783

Table 2: Mass spectrum for $\gamma(b\bar{b})$ system (GeV)

Meson	Experiment	(Sharma &Fiase, 2003)	(Jona, 1983)	This work
$\gamma(1s)$	9.46037$\pm$ 0.21	9.46	9.4336	9.9252
$\gamma(2s)$	10.0233 $\pm$ 0.31	10.0235	9.9944	9.9660
$\gamma(3s)$	10.3553 $\pm$ 0.5	10.3426	10.3230	10.0434
$\gamma(4s)$	10.580 $\pm$ 3.5	10.5695	10.5581	10.1564
$\gamma(5s)$		10.7468	10.7418	10.3039

Table 2 contains the mass spectrum of the bottomium $(b\bar{b})$ system in GeV. The result for this system compares well with experimental results and other works. However, there are some variations at 4s and 5s orbitals where the result in this work is slightly different.

Table 3: Mass spectrum for $\boldsymbol{\Psi}(c\bar{c})$ system (GeV)

Meson	Experiment	(Sharma &Fiase, 2003)	(Jona, 1983)	This work
$\psi(\mathbf{1}s)$	3.097± 0.20	3. 097	3.0672	3.9715
$\psi(\mathbf{2}s)$	3.686 ± 0.03	3.6867	3.6620	4.0222
$\psi(\mathbf{3}s)$	4.030 ± 0.05	4.0398	4.0096	4.1151
$\psi(\mathbf{4}s)$	4.415 ± 0.60	4.3047	4.2579	4.2476
$\psi(\mathbf{5}s)$	4.417 ± 0.01	4.5181	4.4952	4.4161

The mass spectrum for charmonium $(c\bar{c})$ system in Table 3, the 1s orbital is significantly different when compared with the experimental result and other works. However, other orbits compared well.

Table 4: Mass spectrum for $\boldsymbol{\rho}(u\bar{u})$ system (GeV)

Meson	Experiment	(Jona, 1983)	This work
$\boldsymbol{\rho}(\mathbf{1}s)$	0.7685± 0.21	0.770	0.7002
$\boldsymbol{\rho}(\mathbf{2}s)$	1.600	1.402	1.4136
$\boldsymbol{\rho}(\mathbf{3}s)$	_	1.772	2.1821

The up quark shown in Table 4 compared well with Jena, 1983.

Table 5: Mass spectrum for $\emptyset(s\bar{s})$ system (GeV)

Meson	Experiment	(Sharma &Fiase, 2003)	(Jona, 1983)	This work
$\emptyset(\mathbf{1}s)$	1.096± 0.21	1.020	1.004	1.1382
$\emptyset(\mathbf{2}s)$	1.65	1.8352	1.5532	1.3712
$\emptyset(\mathbf{3}s)$	1.90	2.3669	1.8804	2.2067

The result of the strange quark is shown in Table 5 and there is no significant different.

Table 6: Mass spectrum for $(c\bar{u})$ system (GeV)

Meson system	Experiment	(Jona, 1983)	This works
$c\bar{u}(\mathbf{1}s)$	2.010± 0.5	1.9716	1.7065
$c\bar{u}(\mathbf{2}s)$	-	-	1.9228
$c\bar{u}(\mathbf{3}s)$	-	-	2.2605
$c\bar{u}(\mathbf{4}s)$	-	-	2.6705
$c\bar{u}(\mathbf{5}s)$	-	-	3.1228

The result for the mixed quark shown in Table 6 also compared well.

Discussion

This work employed the Dirac equation and PSLET expansion technique to derive a Schrodinger-like equation. A non-QCD based power law potential was then solved in the Schrodinger equation to obtain the eigen- values of Dirac equation as shown in the Table 2-5 above. The mass spectrum for bottom quark, charm quark, up quark, strange quark and mixed quark agreed with experimental data, Sharma and Fiase, 2003 and Jena, 1983. There are however, some discrepancies as observed 1s, 4s and 5s Table 2, 1s in Table 3 and 2s Table5. These variations may be due the fact that only first energy correction was considered in this work.

CONCLUSION

This work has established that a non-QCD based power law potential developed by Sharma and Fiase as discussed earlier can effectively use in the PSLET expansion technique to mimic the spectra of all meson systems. The mass spectrum of meson was calculated implementing the non-columbic power law potential in the Dirac equation. Regardless of the nature of the non-columbic potential which is quite uncomfortable and thus contradicts the predictions of the QCD, the results were still in agreement with experimental results and existing works.

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