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The Direct Nuclear Charge Radii Measurement from Potential Wavefunction

*Adamu, A., Ngadda, Y. H., Hassan, M.

Department of Physics, University of Maiduguri, P. M. B. 1069, Maiduguri, Nigeria

*Corresponding author's email: aliyuphysics@unimaid.edu.ng

ABSTRACT
The charge distribution is a fundamental property of atomic nuclei that affects many of their static characteristics, such as spin, parity, binding energy and effective interaction. Accurately measuring the nuclear charge distribution is of immense importance, as it represents the overall characteristics of the entire nucleus. In this study, we aimed to model a simple and effective formula that could best describe the size of nuclei using the wavefunctions of potential fields originating from positively charged protons. We analyzed finite-size nuclear potential to derive a simple and effective Z-dependent formula. This formula was then applied to calculate the size of various nuclei, and the results were compared with the experimentally measured R and R_{rms}, showing agreement with a deviation of $\delta^2 = 0.0041$. In contrast to the commonly used $A^{1/3}$-dependent formula, our $Z^{1/3}$-dependent formula keeps the radius parameter r_Z almost constant within $4 < Z < 104$. The potential radii of nuclei are reflective of their charge distributions, coulomb energy and lepton energy levels. Additionally, while the $A^{1/3}$-dependent radius measures the matter radius of nuclei, the $Z^{1/3}$-dependent radius, R_Z and $R_{Z,N}$, measures the proton wavefunctions beyond the nucleons boundary. Thus, our study provides a simple and effective Z-dependent formula for describing the size of nuclei that can be used to improve our understanding of nuclear physics and the behavior of atomic nuclei. It also provides another dimension for nuclear size measurements and the range of nuclear-lepton interactions.

Keywords:
Finite-size nucleus,
Nuclear potential charge,
Constant nuclear parameter,
Z-dependence.

INTRODUCTION

The charge distribution of a nucleus, typically characterized by its charge radius, is a fundamental static property that affects many of its other characteristics, such as spin, parity, binding energy and effective interaction (Rahaman *et al*., 2020; Ozawa and Tanihata, 2001; Angeli, 2004; Angeli and Marinova, 2013). Accurately measuring the nuclear size using both theoretical and experimental methods is crucial for testing theoretical models and investigating phenomena such as shape coexistence, shape transition, shell evolution, coulomb potential, nucleon density, single particle orbitals and wavefunctions. The nuclear volume properties are also connected to exotic phenomena such as nuclear halo and neutron skin effect, making them important input quantities for nuclear physics, astrophysics and other theoretical studies (Marinova, 2015; An *et al*., 2021*a*; Sun *et al*., 2015; Thakur and Dhiman, 2019). Therefore, the nuclear size represents the overall characteristics of the entire nucleus and any developments in its measurement techniques can improve our understanding of its structure and complex dynamism (Tel *et al*., 2008; Wang and Li, 2013; An *et al*., 2021*b*).

Experimental evidence supports that the distribution of nuclear matter (protons and neutrons) is approximately uniform, leading to a constant number of nucleons per unit volume. The size of a nucleus, for a constant density $\rho_0 \simeq 0.15\ fm^{-3}$, can be described by the empirical law (Ghoshal, 2007; Basdevant *et al*., 2005; Mohr *et al*., 2016; Utama *et al*., 2016): $R = r_0\,A^{1/3}$, where $r_0 = (3/4\pi\rho_0)^{1/3}$ is an approximation of the range of nuclear forces. However, the parameter r_0 is not constant, especially for nuclei with a proton to neutron ratio less than one. Factors contributing to the discrepancy in measuring R include the dependence of the $A^{1/3}$ formula on nuclear isospin, nuclear deformation, nuclear moments, and nucleon pairing energy (Royer, 2008; Royer and Rousseau, 2019; Merino *et al*., 2009; Patoary and Oreshkina, 2017). In light of these factors, there have been improvements to the $A^{1/3}$ dependence formula. These improvements include incorporating corrections such as surface effects (Wakasa *et al*., 2022), isotopic shifts (Nerlo-Pomorska and Pomorski, 1993), isospin

(Nerlo-Pomorska and Pomorski, 1994), shell corrections (Bao *et al*., 2020) for exotic nuclei far from the β-stability line (Yi-An *et al*., 2009), halo nuclei (Antonov *et al*., 2005), odd-even staggering effects (Sheng et al., 2015; Talmi, 1984), finite-size nucleus (Adamu and Ngadda, 2017; Nickel, 2013), volume effect, and nuclear deformations (Reinhard and Nazarewicz, 2022). Royer and Rousseau (2019) introduced an N-dependence formula to account for the influence of deformation, moments, and nucleon pairing energy. In addition, $Z^{1/3}$ and $N^{1/3}$ dependence formulas have been proposed to obtain a constant nuclear parameter (Sun *et al*., 2015; Bayram *et al*., 2013).

The issue of nuclear sizes is characterized by radial and angular variations of matter, charge, and effective interactions. The matter distribution is represented by the radius, R, while the charge distribution is represented by the radius $\langle r^2\rangle^{1/2}$. These quantities provide valuable insights into nuclear potentials and wavefunctions (Merino *et. al*., 2009; Patoary and Oreshkina, 2017; Akkoyun *et. al*., 2013), and are related to the non-uniform radius parameter r_0. Various physical quantities in nuclear physics and related fields depend on R, and thus r_0, but no single formula incorporating both experimental and theoretical data can provide a constant value for r_0. This poses a significant challenge for nuclear physicists, suggesting that some physics may be missing in the current formula (Bethe, 1936; Reinhard and Nazarewicz, 2022). In this study, we propose a new simple analytic relation that links the potential radius with the proton charge distribution, allowing us to obtain a constant nuclear parameter by considering the fact that the nuclear potential is dependent only on the proton charges, $+Ze$.

MATERIALS AND METHODS

In an attempt to getting the constant value of r_0, Ref. (Adamu and Ngadda, 2017) considered nuclear-lepton effective interaction and come up with the formula:

$$R_p = \sqrt{3}R \qquad (1)$$

which tends to measure the nuclear potential radii. To support this result, analyses have been made from the study of β^+-decay energy of finite-size nucleus and obtained the formula (Adamu, 2021):

$$R_C = \sqrt{3}R \qquad (2)$$

which agreed with the relation (1). However, the radius potential R_p and R_C still did not provide promising results as it indirectly depends on non-constant r_0. Although efforts are made to successfully describe general trends of charge radii using various nuclear models, an accurate description of charge radii and their local variations remains a major challenge (Xie *et. al.*, 2019; Ma and Zhang, 2022). The present study considers the atomic nucleus as having a spherically symmetric charge distribution $\rho(r) = 3Ze/4\pi R^3$ and the effective interaction is described by the lepton-nuclear potential energy:

$$U(\vec{r}, R) = -\frac{Z\gamma}{2r}\left[\frac{3r}{R} - \left(\frac{r}{R}\right)^3\right] \qquad (3)$$

where $\gamma = ke^2$ is the coulomb constant and $R = r_0A^{1/3}$, with $r_0 = 1.2 \times 10^{-15}m$. The potential (3) described the interaction of lepton with nucleus having a finite-size charge distribution. The potential (3) gives the reasonable analytical behavior,

$$U(\vec{r}) = -\frac{Z\gamma}{2R}$$

of the lepton wavefunctions at $r = 0$, by modifying the infinite value of the conventional coulomb potential as $r \to 0$ (Michel and Oreshkina, 2021; Oreshkina, 2022; Shahaev, 1993; Kumar *et. al*., 2021; Niri and Anjami, 2018; Deck *et. al*., 2005; Neznamov and Safronov, 2004):

$$U(\vec{r}) = -\frac{Z\gamma}{r} \qquad (4)$$

In the same manner, this study uses the numerical values computed from potential (3) to determine the radii that fits the nuclear size measurements for atomic nucleus and for electrodynamics and astronomical bodies composing of multiple $\pm Ze$ charges, where Z represents the number of lepton charges.

RESULTS AND DISCUSSION

Results

The present study is aimed to determine the Z-dependence formula for nuclear potential radii with constant parameter, r_Z, using the effective lepton-nuclear interaction (3). The Z-dependent form of nuclear charge radii is obtained when the relation $R_p = r_Z Z^n$ was considered for fitting. This gives the potential radius of the form:

$$R_p = 1.5Z^{1/3} \qquad (5)$$

The relation (5) described the radius of lepton ($\pm e$) charge distribution, which can be applied to determine the size of physical bodies (for example quarks, leptons) composing of multiple $\pm Ze$ charges, where Z represents the number of lepton charges and e is the unit or fraction (in case of quarks) of electric charge. To measure the size of atomic nucleus where proton and neutron interaction via strong and electrostatic force, the relation (5) has to be improved by adding terms to the parameter, r_Z as follows:

$$R_{Z,N} = r_Z Z^{1/3} + \alpha Z + \beta \qquad (6)$$

where the parameter r_Z = 1.500 *fm*, α = 0.007 and β = -0.045. Table 1 showed the numerical values of R_Z, the improved nuclear potential radii, $R_{Z,N}$ and the conventional A-dependent nuclear radius: $R = r_0A^{1/3}$. When the relation (6) is computed for various nuclei, the results are found to agree with the experimentally measured R and R_{rms} as shown in Table 1.

Table 1: The relationship between R and $R_{Z,N}$

$^{A}X_{Z}$	Nuclear Size Measurements for Nuclides ($^{A}X_{Z}$)				$^{A}X_{Z}$	Nuclear Size Measurements for Nuclides ($^{A}X_{Z}$)				
	R	R_p	$R_{Z,N}$	δR_Z	δR		R	R_p	$R_{Z,N}$	δR_Z	δR
$^{1}H_{1}$	1.2000	1.5000	1.4620	0.3000	-0.2620	$^{133}Cs_{55}$	6.1254	5.7044	6.0444	-0.4210	0.0810
$^{4}He_{2}$	1.9049	1.8899	1.8589	-0.0150	0.0460	$^{130}Ba_{56}$	6.0790	5.7388	6.0858	-0.3402	-0.0068
$^{6}Li_{3}$	1.9049	2.1634	2.1394	0.2585	-0.2345	$^{139}La_{57}$	6.2161	5.7728	6.1268	-0.4433	0.0893
$^{9}Be_{4}$	2.4961	2.3811	2.3641	-0.1150	0.1320	$^{136}Ce_{58}$	6.1711	5.8063	6.1673	-0.3648	0.0038
$^{10}B_{5}$	2.5853	2.5650	2.5550	-0.0203	0.0303	$^{141}Pr_{59}$	6.2458	5.8395	6.2075	-0.4063	0.0383
$^{12}C_{6}$	2.7473	2.7257	2.7227	-0.0216	0.0246	$^{142}Nd_{60}$	6.2605	5.8723	6.2473	-0.3882	0.0132
$^{14}N_{7}$	2.8922	2.8694	2.8734	-0.0228	0.0188	$^{145}Pm_{61}$	6.3043	5.9047	6.2867	-0.3996	0.0176
$^{16}O_{8}$	3.0238	3.0000	3.0110	-0.0238	0.0128	$^{144}Sm_{62}$	6.2898	5.9368	6.3258	-0.3530	-0.0360
$^{19}F_{9}$	3.2021	3.1201	3.1381	-0.0820	0.0640	$^{151}Eu_{63}$	6.3901	5.9686	6.3646	-0.4215	0.0255
$^{20}Ne_{10}$	3.2573	3.2317	3.2567	-0.0256	0.0006	$^{154}Gd_{64}$	6.4321	6.0000	6.4030	-0.4321	0.0291
$^{23}Na_{11}$	3.4126	3.3360	3.3680	-0.0766	0.0446	$^{159}Tb_{65}$	6.5010	6.0311	6.4411	-0.4699	0.0599
$^{24}Mg_{12}$	3.4614	3.4341	3.4731	-0.0273	-0.0117	$^{156}Dy_{66}$	6.4599	6.0619	6.4789	-0.3980	-0.0190
$^{27}Al_{13}$	3.6000	3.5270	3.5730	-0.0730	0.0270	$^{165}Ho_{67}$	6.5818	6.0923	6.5163	-0.4895	0.0655
$^{28}Si_{14}$	3.6439	3.6152	3.6682	-0.0287	-0.0243	$^{162}Er_{68}$	6.5416	6.1225	6.5535	-0.4191	-0.0119
$^{31}P_{15}$	3.7697	3.6993	3.7593	-0.0704	0.0104	$^{169}Tm_{69}$	6.6345	6.1523	6.5903	-0.4822	0.0442
$^{32}S_{16}$	3.8098	3.7798	3.8468	-0.0300	-0.0370	$^{168}Yb_{70}$	6.6214	6.1819	6.6269	-0.4395	-0.0055
$^{35}Cl_{17}$	3.9253	3.8569	3.9309	-0.0684	-0.0056	$^{175}Lu_{71}$	6.7121	6.2112	6.6632	-0.5009	0.0489
$^{36}Ar_{18}$	3.9623	3.9311	4.0121	-0.0312	-0.0498	$^{176}Hf_{72}$	6.7249	6.2403	6.6993	-0.4846	0.0256
$^{39}K_{19}$	4.0695	4.0026	4.0906	-0.0669	-0.0211	$^{181}Ta_{73}$	6.7880	6.2690	6.7350	-0.5190	0.0530
$^{40}Ca_{20}$	4.1039	4.0716	4.1666	-0.0323	-0.0627	$^{180}W_{74}$	6.7755	6.2975	6.7705	-0.4780	0.0050
$^{45}Sc_{21}$	4.2683	4.1384	4.2404	-0.1299	0.0279	$^{185}Re_{75}$	6.8376	6.3257	6.8057	-0.5119	0.0319
$^{46}Ti_{22}$	4.2997	4.2031	4.3121	-0.0966	-0.0124	$^{184}Os_{76}$	6.8253	6.3537	6.8407	-0.4716	-0.0154
$^{51}V_{23}$	4.4501	4.2658	4.3818	-0.1843	0.0683	$^{191}Ir_{77}$	6.9108	6.3815	6.8755	-0.5293	0.0353
$^{50}Cr_{24}$	4.4208	4.3267	4.4497	-0.0941	-0.0289	$^{192}Pt_{78}$	6.9228	6.4090	6.9100	-0.5138	0.0128
$^{55}Mn_{25}$	4.5635	4.3860	4.5160	-0.1775	0.0475	$^{197}Au_{79}$	6.9824	6.4363	6.9443	-0.5461	0.0381
$^{54}Fe_{26}$	4.5357	4.4437	4.5807	-0.0920	-0.0450	$^{196}Hg_{80}$	6.9705	6.4633	6.9783	-0.5072	-0.0078
$^{59}Co_{27}$	4.6716	4.5000	4.6440	-0.1716	0.0276	$^{203}Tl_{81}$	7.0526	6.4901	7.0121	-0.5625	0.0405
$^{58}Ni_{28}$	4.6451	4.5549	4.7059	-0.0902	-0.0608	$^{204}Pb_{82}$	7.0641	6.5167	7.0457	-0.5474	0.0184
$^{63}Cu_{29}$	4.7749	4.6085	4.7665	-0.1664	0.0084	$^{209}Bi_{83}$	7.1214	6.5431	7.0791	-0.5783	0.0423
$^{64}Zn_{30}$	4.8000	4.6608	4.8258	-0.1392	-0.0258	$^{209}Po_{84}$	7.1214	6.5693	7.1123	-0.5521	0.0091
$^{69}Ga_{31}$	4.9219	4.7121	4.8841	-0.2098	0.0378	$^{209}At_{85}$	7.1214	6.5952	7.1452	-0.5262	-0.0238
$^{70}Ge_{32}$	4.9455	4.7622	4.9412	-0.1833	0.0043	$^{222}Rn_{86}$	7.2661	6.6210	7.1780	-0.6451	0.0881
$^{75}As_{33}$	5.0606	4.8113	4.9973	-0.2493	0.0633	$^{223}Fr_{87}$	7.2770	6.6466	7.2106	-0.6304	0.0664
$^{74}Se_{34}$	5.0380	4.8594	5.0524	-0.1786	-0.0144	$^{226}Ra_{88}$	7.3094	6.6719	7.2429	-0.6375	0.0665
$^{79}Br_{45}$	5.1490	4.9066	5.1066	-0.2424	0.0424	$^{227}Ac_{89}$	7.3202	6.6971	7.2751	-0.6231	0.0451
$^{78}Kr_{36}$	5.1272	4.9529	5.1599	-0.1743	-0.0327	$^{232}Th_{90}$	7.3736	6.7221	7.3071	-0.6515	0.0665
$^{85}Rb_{37}$	5.2762	4.9983	5.2123	-0.2779	0.0639	$^{231}Pa_{91}$	7.3630	6.7469	7.3389	-0.6161	0.0241
$^{84}Sr_{38}$	5.2554	5.0430	5.2640	-0.2124	-0.0086	$^{238}U_{92}$	7.4366	6.7715	7.3705	-0.6651	0.0661
$^{89}Y_{39}$	5.3577	5.0868	5.3148	-0.2709	0.0429	$^{237}Np_{93}$	7.4262	6.7960	7.4020	-0.6302	0.0242
$^{90}Zr_{40}$	5.3777	5.1299	5.3649	-0.2478	0.0128	$^{244}Pu_{94}$	7.4986	6.8203	7.4333	-0.6783	0.0653
$^{93}Nb_{41}$	5.4368	5.1723	5.4143	-0.2645	0.0225	$^{243}Am_{95}$	7.4883	6.8444	7.4644	-0.6439	0.0239
$^{92}Mo_{42}$	5.4172	5.2140	5.4630	-0.2032	-0.0458	$^{247}Cm_{96}$	7.5292	6.8683	7.4953	-0.6609	0.0339

$^{98}Tc_{43}$	5.5325	5.2551	5.5111	-0.2774	0.0214	$^{247}Bk_{97}$	7.5292	6.8921	7.5261	-0.6371	0.0031
$^{100}Ru_{44}$	5.5699	5.2955	5.5585	-0.2744	0.0114	$^{251}Cf_{98}$	7.5696	6.9157	7.5567	-0.6539	0.0129
$^{103}Rh_{45}$	5.6251	5.3353	5.6053	-0.2898	0.0198	$^{252}Es_{99}$	7.5796	6.9391	7.5871	-0.6405	-0.0075
$^{102}Pd_{46}$	5.6068	5.3746	5.6516	-0.2322	-0.0448	$^{257}Fm_{100}$	7.6294	6.9624	7.6174	-0.6670	0.0120
$^{107}Ag_{47}$	5.6970	5.4132	5.6972	-0.2838	-0.0002	$^{260}Md_{101}$	7.6590	6.9855	7.6475	-0.6735	0.0115
$^{106}Cd_{48}$	5.6791	5.4514	5.7424	-0.2277	-0.0633	$^{262}Lw_{102}$	7.6786	7.0085	7.6775	-0.6701	0.0011
$^{113}In_{49}$	5.8015	5.4890	5.7870	-0.3125	0.0145	$^{261}Rf_{103}$	7.6688	7.0313	7.7073	-0.6375	-0.0385
$^{112}Sn_{50}$	5.7843	5.5260	5.8310	-0.2583	-0.0467	$^{262}Db_{104}$	7.6786	7.0540	7.7370	-0.6246	-0.0584
$^{121}Sb_{51}$	5.9353	5.5626	5.8746	-0.3727	0.0607	$^{263}Sg_{105}$	7.6884	7.0765	7.7665	-0.6119	-0.0781
$^{120}Te_{52}$	5.9189	5.5988	5.9178	-0.3201	0.0011	$^{262}Ns_{106}$	7.6786	7.0989	7.7959	-0.5797	-0.1173
$^{127}I_{53}$	6.0318	5.6344	5.9604	-0.3974	0.0714	$^{264}Hs_{107}$	7.6981	7.1212	7.8252	-0.5769	-0.1271
$^{124}Xe_{54}$	5.9840	5.6696	6.0026	-0.3144	-0.0186	$^{266}Mt_{108}$	7.7175	7.1433	7.8543	-0.5742	-0.1368

To examine the validity of the formula (6), the root-mean-square deviation (Royer and Gautier, 2006):

$$\delta^2 = \sum_{109}^{n=1}\left[R - R_{Z,N}\right]^2/n \qquad (7)$$

has been applied and obtain the value of $\delta^2 = 0.0041$. This indicates that the formula (7) is in agreement with a conventional A-dependent formula R. Thus, the formula obtained from this study based on nuclear finite-size gives much better description of nuclear potential charge distribution than the formula (1) and (2). Figure 1 showed the difference between experimental R and our new model, $R_{Z,N}$.

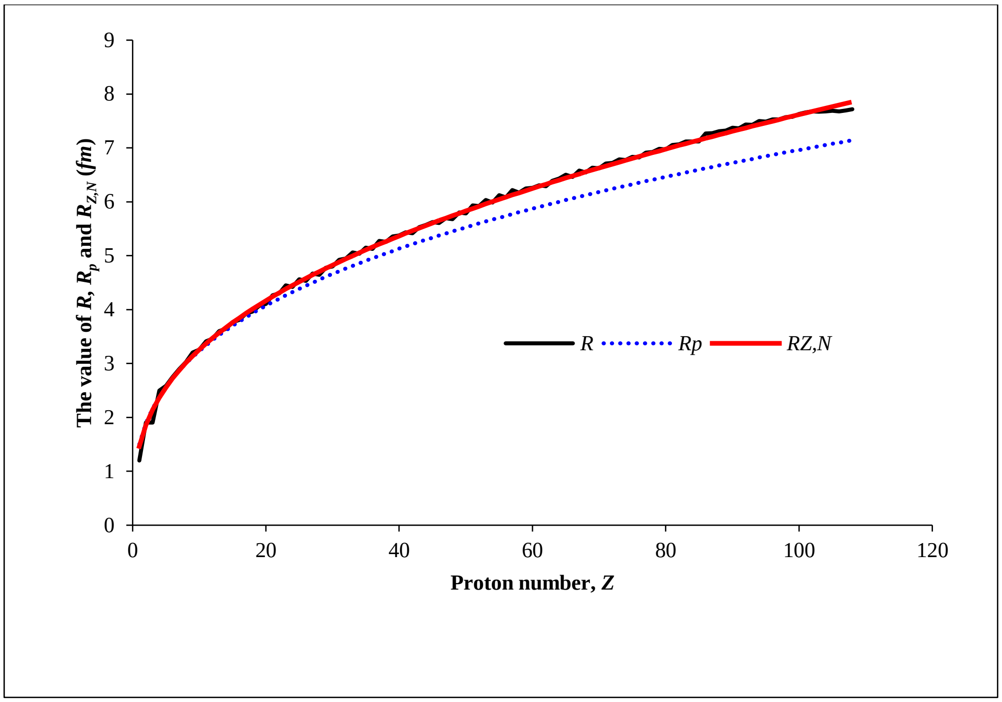

Figure 1: The value of R, R_p and $R_{Z,N}$ (fm) as a function of Z

Figure 1 showed that the potential radius decreases with increase in the proton number. This is because as proton number increases, the number of neutron inside the nucleus starts to outnumber protons and thus affects the nuclear potential. The rough variation of the potential radii $R_{Z,N}$ and the conventional A-dependent nuclear radius R are presented on Figure 2.

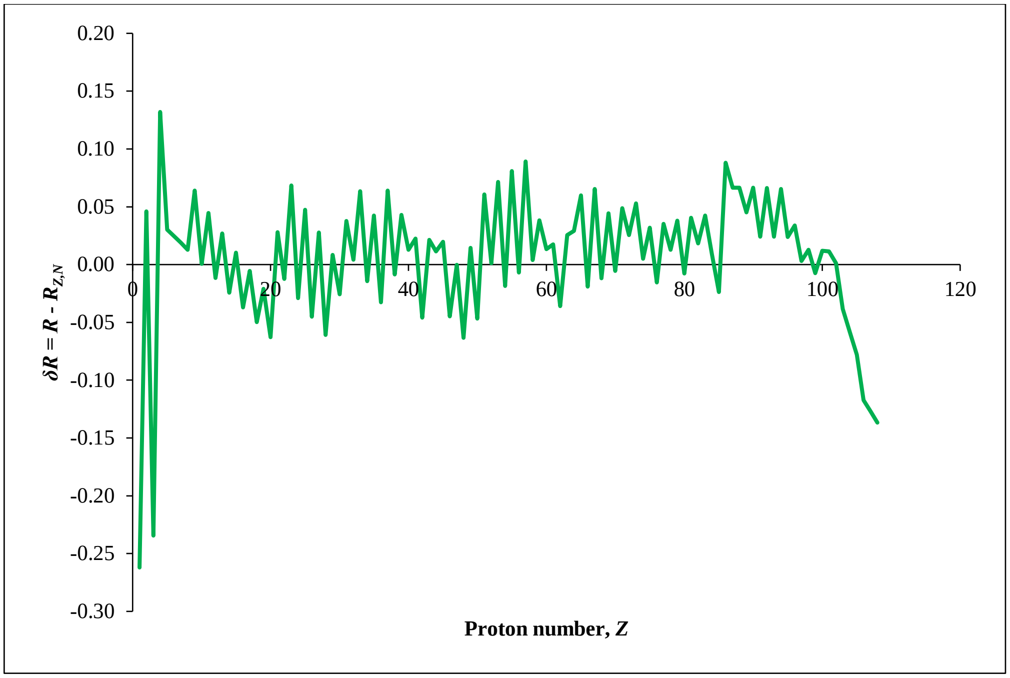

Figure 2: The uneven variation of $\delta R = R - R_{Z,N}$ as a function of Z

It can be observed from Figure 2 the difference between the improved nuclear potential radii $R_{Z,N}$ and the conventional A-dependent nuclear radius R is between -0.26 *fm* and 0.13 *fm*. The significant differences are found on Z < 6 and Z > 104 nuclei. The mean value are 0.0034 *fm* and 0.0619 *fm* respectively. There are eight places with zero value difference and the difference ±0.01 appeared most frequently which is the value for median nuclei.

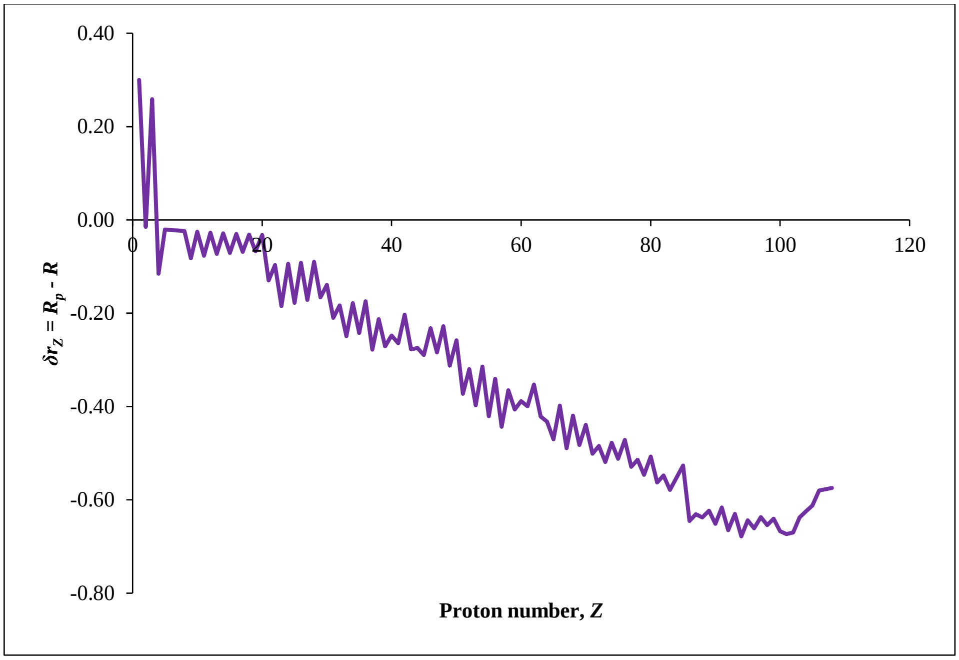

Figure 3: The uneven variation of $\delta r_Z = R_p - R$ as a function of Z

It can be seen from Figure 3 that the variation of potential and nuclear radius decreases with proton number. For hydrogen with only proton nucleus, the potential radius is found to be greater than nuclear radii. This showed that the potential radii decrease when the neutrons are present.

Discussion

The comparison of the new potential radius parameters and the parameters with their corresponding *rms* deviations obtained by previous phenomenological nuclear charge radius formulas obtained by Ref. (Ma and Zhang, 2022) is shown in Table 2. The table presents the values of the parameters r_Z, α, and β for the present study and for the formula obtained by Ma and Zhang, as well as the corresponding values of the root-mean-square (*rms*) deviations for each formula.

Table 2: The parameters and the root-mean-square deviations (δ^2) for the some phenomenological nuclear charge radius formulae

Formula	Parameters	δ^2 (*fm*)
$R_C = r_A A^{1/3}$	r_A = 1.223 *fm* (Bohr and Mottelson, 1975)	0.094
$R_C = r_N N^{1/3}$	r_N = 1.472 *fm* (Bayram *et. al.*, 2013)	0.151
$R_C = r_Z Z^{1/3}$	r_Z = 1.631 *fm* (Reinhard and Nazarewicz, 2021)	0.076
$R_C = r_A\left[1 - \frac{b(N-Z)}{A}\right]A^{1/3}$	r_A = 1.269 *fm*; *b* = 0.252 (Nerlo-Pomorska and Pomorski, 1993)	0.068
$R_C = r_N\left[1 - \frac{b(N-Z)}{N}\right]N^{1/3}$	r_N = 1.629 *fm*; *b* = 0.451 (Bayram *et. al.*, 2013)	0.063
$R_C = r_Z\left(1 + \frac{5}{8}\right)\left[1 + \frac{b(N-Z)}{Z}\right]Z^{1/3}$	r_Z = 1.631 *fm*; *b* = 0.062 (Myers and Swiatecki, 1998)	0.057
$R_C = r_A A^{1/3}$	r_A = 1.227 *fm* (Ma and Zhang, 2022)	0.093
$R_C = r_N N^{1/3}$	r_N = 1.470 *fm* (Ma and Zhang, 2022)	0.151
$R_C = r_Z Z^{1/3}$	r_Z = 1.639 *fm* (Ma and Zhang, 2022)	0.072
$R_C = r_A\left[1 - \frac{b(N-Z)}{A}\right]A^{1/3}$	r_A = 1.282 *fm*; *b* = 0.342 (Ma and Zhang, 2022)	0.065
$R_C = r_N\left[1 - \frac{b(N-Z)}{N}\right]N^{1/3}$	r_N = 1.623 *fm*; *b* = 0.438 (Ma and Zhang, 2022)	0.063
$R_C = r_Z\left(1 + \frac{5}{8}\right)\left[1 + \frac{b(N-Z)}{Z}\right]Z^{1/3}$	r_Z = 1.634 *fm*; *b* = 0.220 (Ma and Zhang, 2022)	0.049
$R_p = r_Z Z^{1/3}$	r_Z = 1.500 *fm* (Present study)	-
$R_{Z,N} = r_Z Z^{1/3} + \alpha Z - \beta$	r_Z = 1.500 *fm*; α = 0.07; β = 0.045 (Present study)	0.004

Table 2 presented in the study reveals that the new potential radius formula developed in this research yields a slightly larger value of r_Z compared to the formula derived by Ma and Zhang (2022). Furthermore, the parameters α and β also differ between the two formulas. The root-mean-square (*rms*) deviation for the formula proposed in this study is smaller than that of the formula obtained by Ma and Zhang, indicating that the new potential radius formula fits experimental data better than the previous one. The table demonstrates that utilizing the improved parameter $R_{Z,N}$ leads to a slight reduction in *rms* deviations compared to the traditional A-dependent nuclear radius R. It is crucial to acknowledge that the potential radius R_p and the nuclear radius $R_{Z,N}$ measure different aspects of the nucleus and should not be directly compared. R_p represents the potential charge radii, which indicates the distance from the nucleus center to the average position of the positive charge of the protons. On the other hand, $R_{Z,N}$ quantifies the size of the atomic nucleus, where protons and neutrons interact via strong and electrostatic forces. However, it is worth noting that the one-parameter potential radii achieve accuracy comparable to $R_{Z,N}$, representing an improvement over the conventional A-dependent formula *R*. This suggests that the new Z-dependence formula, incorporating the constant parameter r_Z and utilizing the effective lepton-nuclear interaction, provides a better approximation for determining the size of the atomic nucleus. The introduction of parameters α and β in equation (6) aims to account for the non-perfectly spherical shape of the nucleus, which varies due to the proton configuration. Different nuclear configurations, such as cigar-like, triangular, or square-like charge distributions, arise due to the repulsive forces between positively charged protons. However, the strong force between protons and neutrons can also influence the possible configurations, especially in light nuclei where the strong force can overcome the repulsive Coulomb force.

By incorporating nuclear finite-size models into theoretical predictions, researchers can make testable theoretical predictions concerning nuclear and atomic parameters. The Z-dependence formula for nuclear potential radii, utilizing the constant parameter r_Z and effective lepton-nuclear interaction, serves as a valuable tool for determining the atomic nucleus's size. The inclusion of α and β terms allows for accounting for the variation in nuclear shape resulting from proton configuration. This formula represents an improvement over the conventional A-dependent formula and achieves comparable accuracy to the nuclear radius $R_{Z,N}$.

Therefore, the utilization of theoretical models enhances our understanding of nuclear and atomic properties, guiding future experimental efforts.

CONCLUSION

Indeed, the $Z^{1/3}$-dependent formula proposed in the present study offers a simpler and more effective approach to calculating nuclear potential radii and other charged bodies. The formula's ability to maintain a nearly constant value of radius parameter r_p within a wide range of atomic numbers ($4 < Z < 104$) makes it a valuable tool for understanding nuclear size and the range of nuclear-lepton interaction. This new dimension of measuring nuclear size has the potential to advance research in nuclear and atomic physics, and could lead to improved theoretical predictions that agree more closely with experimental results.

RECOMMENDATIONS

This study proposes a simple and effective formula based on wavefunctions originating from positively charged protons to describe the size of nuclei. The authors recommend further research in the following areas:

i. Conduct experimental measurements of the nuclear charge distribution using advanced techniques such as electron scattering or muonic atoms.
ii. Investigate the relationship between isotopes of the same element and the formula by analyzing size variations among them.
iii. Examine the impact of nuclear structure effects, such as deformation and shape variation, on nuclear sizes using the formula.
iv. Develop a theoretical framework to explain the underlying physics of the Z-dependent formula, focusing on the connections between wavefunctions, charge distribution, and factors like Coulomb energy and lepton energy levels.
v. Explore the extension of the formula to understand properties beyond atomic nuclei.

By pursuing these research directions, scientists can advance our understanding of nuclear physics, improve the accuracy of measuring nuclear sizes, and gain new insights into atomic nuclei and their interactions.

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