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APPLICABILITY OF HULTHÉN-HELLMANN POTENTIAL TO PREDICT THE MASS-SPECTRA OF HEAVY MESONS VIA SERIES EXPANSION METHOD

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ABSTRACT

Hulthén plus Hellmann potential was adopted as the quark-antiquark interaction potential for predicting the mass spectra of heavy mesons. The adopted potential was made to be temperature-dependent by replacing the screening parameter with Debye mass $\left(m_D(T)\right)$. The radial Schrödinger equation was analytically solved using the series expansion method and energy eigenvalues were obtained. The energy eigenvalues was used to predict the mass spectra of heavy mesons such as charmonium $(c\bar{c})$ and bottomonium $(b\bar{b})$. Four special cases were considered when some of the potential parameters were set to zero, resulting in Hellmann potential, Yukawa potential, Coulomb potential, and Hulthén potential, respectively. The present potential provides satisfying results in comparison with experimental data and work of other researchers with a maximum error of 0.034 GeV.

Keywords: Hulthén potential, Hellmann potential, Heavy mesons, Series expansion method.

INTRODUCTION

The solution of the Schrödinger equation (SE) for a physical system in quantum mechanics is of great importance, because the knowledge of Eigen energy and wave function contains all possible information about the physical properties of a system under study (Inyang *et al*., 2021). The study of behavior of several physical problems in physics requires us to solve the non-relativistic or relativistic equation. A good description of many features of these problems can be obtained using non-relativistic models that is the quark-antiquark strong interaction is described by a phenomenological potential (Abu-shady *et al.*, 2021). Heavy mesons have turned out to provide extremely useful probes for the deconfined state of matter because the force between a heavy quark and anti-quark is weakened due to the presence of gluons which lead to the dissociation of it bound states **(**Allosh *et al.*, 2021). The heavy mesons and their interaction are well described by the SE (Prasanth *et al*., 2020; Inyang *et al.,* 2021).

The solution of the spectral problem for the SE with spherically symmetric potentials is of major concern in describing the spectra of heavy mesons(Rani *et al*.,2018). Potential models offer a rather good description of the mass spectra of heavy mesons such as bottomonium, and charmonium (Mansour and Gamal, 2018). In predicting the mass spectra of heavy mesons, confining-type potentials are generally used. The holding potential is the Cornell potential with two terms one of which is responsible for the Coulomb interaction of the quarks and the other corresponds to a confining term (Al-Jamel, 2019). In the past, this type of potential has been studied by many researchers using different techniques. For instance, Al-Jamel (2019) studied the mass spectra with Cornell potential using the asymptotic iteration method (AIM). Abu-Shady, (2016) solved the SE with Cornell potential using the Nikiforov-Uvarov (NU) method. The obtained energy equation were used to study the mass spectra of heavy mesons. In addition, Ciftci, and Kisoglu, (2018) obtained the solutions of SE with Cornell potential using the AIM. The masses of the heavy mesons such charmonium and bottomonium were studied. The confining potentials may be in different forms depending upon the interaction of the particles within the system. Harmonic oscillator and hydrogen atom are the two potentials which solutions to the SE are found exactly. There are several approaches to obtaining such approximate solutions. For instance, the asymptotic iteration method (AIM) as carried out by Al-Jamel, (2019), the Laplace transformation method as used by Abu-Shady and Khokha, (2018), the super symmetric quantum mechanics (SUSYQM) method as applied by Abu-Shady and Ikot, (2019) and the series expansion method (SEM) used byInyang*et al*. (2021). Other methods include the analytical exact iterative method (AEIM) by Khokha *et al.* (2016), the exact quantization rule (EQR) by Inyang *et al*. (2020) etc.

The Hulthén potential, (1942) is a short-range potential that behaves like a Coulomb potential for small values of r and decreases exponentially for large values of r. It has been used in many branches of Physics, such as Nuclear and Particle Physics, Atomic Physics, Solid-State Physics, and Chemical Physics (Oyewumi and Oluwadare, 2016).

The Hellmann potential,(1935) which is a superposition of an attraction Coulomb potential and a Yukawa potential has been studied extensively by many authors in obtaining the energy of the bound state in atomic, nuclear, and particle physics (William *et al.*, 2020).

Recently, there has been great interest in combining two potentials in both the relativistic and non-relativistic regime (William *et al.*, 2020). The essence of combining two or more physical potential models is to have a wider range of applications. Hence, in the present work, we aim at solving the SE with the combination of Hulthén and Hellmann potential (HHP) analytically using series expansion method and apply the results to predict the mass spectra of heavy mesons such as bottomonium and charmonium, in which the quarks are considered as spinless particles for easiness

The adopted potential is of the form (William *et al.*, 2020)

$$V(r) = -\frac{A_0 e^{-\alpha r}}{1-e^{-\alpha r}} - \frac{A_1}{r} + \frac{A_2 e^{-\alpha r}}{r} \quad (1)$$

where A_0, A_1 and A_2 are potential strength parameters and α is the screening parameter. To make equation (1) temperature dependent, the screening parameter is replaced with Debye mass $(m_D(T))$ which vanishes at T$\to 0$ and we have,

$$V(r,T) = -\frac{A_0 e^{-m_D(T)r}}{1-e^{-m_D(T)r}} - \frac{A_1}{r} - \frac{A_2 e^{-m_D(T)r}}{r} \quad (2)$$

The expansion of the exponential terms in equation (2) (up to order three, in order to model the potential to interact in the quark-antiquark system) yields,

$$\frac{e^{-m_D(T)r}}{r} = \frac{1}{r} - m_D(T) + \frac{m_D^2(\mathrm{T})\mathrm{r}}{2} - \frac{m_D^3(T)r^2}{6} + \ldots \quad (3)$$

$$\frac{e^{-m_D(T)r}}{1-e^{-m_D(T)\mathrm{r}}} = \frac{1}{m_D(T)r} - \frac{1}{2} + \frac{m_D(T)r}{12} + \ldots (4)$$

The substitution of equations (3) and (4) into equation (2) gave equation (5)

$$V(r,T) = -\frac{\beta_0}{r} + \beta_1 r - \beta_2 r^2 + \beta_3 (5)$$

where

$$\left.\begin{aligned} -\beta_0 &= -A_1 + A_2 - \frac{A_0}{m_D(T)}, \ \beta_1 = \frac{A_2 m_D^2(T)}{2} - \frac{A_0 m_D(T)}{12} \\ \beta_2 &= \frac{A_2 m_D^3(T)}{6}, \ \beta_3 = \frac{A_0}{2} - A_2 m_D(T) \end{aligned}\right\} \quad (6)$$

The first term in equation (5) is the Coulomb potential that describes the short distance between quarks, while the second term is a linear term for confinement feature.

SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH HULTHÉN PLUS HELLMANN POTENTIAL

The radial SE was considered (Ibekwe*et al.*, 2021)

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{2\mu}{\hbar^2}\left(E_{nl} - V(r)\right) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \quad (7)$$

Where l is angular quantum number taking the values 0,1,2,3,4…, μ is the reduced mass for the heavy mesons, r is the inter nuclear separation and E_{nl} denotes the energy eigenvalues of the system.

The substitution of equation (5) into equation (7) gives

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\varepsilon + \frac{P}{r} - Qr + Sr^2 - \frac{\mathrm{L(L+1)}}{r^2}\right]R(r) = 0 \quad (8)$$

Where

$$\left.\begin{aligned} \varepsilon &= \frac{2\mu}{\hbar^2}\left(E_{nl} - \beta_3\right), \mathrm{P} = \frac{2\mu\beta_0}{\hbar^2} \\ Q &= \frac{2\mu\beta_1}{\hbar^2}, \mathrm{S} = \frac{2\mu\beta_2}{\hbar^2} \end{aligned}\right\} \quad (9)$$

$$L\left(L+1\right) = l\left(l+1\right) (10)$$

From equation (10)

$$L=-\frac{1}{2}+\frac{1}{2}\sqrt{(2l+1)^2} \quad (11)$$

Now make an anzats wave function (Rani*et al.*, 2018)

$$R(r)=e^{-\alpha r^2-\beta r}F(r) \quad (12)$$

where α and β are positive constants whose values are to be determined in terms of potential parameters. Differentiating equation (12) twice gives

$$R'(\mathrm{r})=F'(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F(r)(-2\alpha r-\beta)e^{-\alpha r^2-\beta r} \quad (13)$$

$$R''(\mathrm{r})=\mathrm{F}''(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F'(r)(-2\alpha r-\beta)e^{-\alpha r^2-\beta r}+\left[(-2\alpha)+(-2\alpha r-\beta)(-2\alpha r-\beta)\right]F(r)e^{-\alpha r^2-\beta r} \quad (14)$$

The substitution of equations (12), (13) and (14) into equation (8) and dividing by $e^{-\alpha r^2-\beta r}$ gives

$$F''(\mathrm{r})+\left[-4\alpha r-2\beta+\frac{2}{r}\right]F'(\mathrm{r})+\left[\begin{array}{l}(4\alpha^2+S)r^2+(4\alpha\beta-B)r\\+(P-2\beta)\frac{1}{r}-\frac{L(L+1)}{r^2}+(\varepsilon+\beta^2-6\alpha)\end{array}\right]F(r)=0 \quad (15)$$

The function $F(r)$ is considered as a series of the form

$$F(r)=\sum_{n=0}^{\infty}a_n r^{2n+L} \quad (16)$$

Taking the first and second derivatives of equation (16) we obtain,

$$F'(r)=\sum_{n=0}^{\infty}(2n+L)a_n r^{2n+L-1} \quad (17)$$

$$F''(r)=\sum_{n=0}^{\infty}(2n+L)(2n+L-1)a_n r^{2n+L-2} \quad (18)$$

Substituting equations (16), (17) and (18) into equation (15), gives

$$\sum_{n=0}^{\infty}(2n+L)(2n+L-1)a_n r^{2n+L-2}+\left[-4\alpha r-2\beta+\frac{2}{r}\right]\sum_{n=0}^{\infty}(2n+L)a_n r^{2n+L-1}+\left[(4\alpha^2+S)r^2+(4\alpha\beta-B)r+(P-2\beta)\frac{1}{r}-\frac{L(L+1)}{r^2}+(\varepsilon+\beta^2-6\alpha)\right]\sum_{n=0}^{\infty}a_n r^{2n+L}=0 \quad (19)$$

By collecting powers of r in equation (19) we have

$$\sum_{n=0}^{\infty}a_n\left\{\begin{array}{l}\left[(2n+L)(2n+L-1)+2(2n+L)-L(L+1)\right]r^{2n+L-2}\\+\left[-2\beta(2n+L)+(P-2\beta)\right]r^{2n+L-1}\\+\left[-4\alpha(2n+L)+\varepsilon+\beta^2-6\alpha\right]r^{2n+L}\\+\left[4\alpha\beta-Q\right]r^{2n+L+1}+\left[4\alpha^2+S\right]r^{2n+L+2}\end{array}\right\}=0 \quad (20)$$

Equation (20) is linearly independent implying that each of the terms is separately equal to Zero, noting that r is a non-zero function; therefore, it is the coefficient of r that is zero. With this, the relation for each of the terms is obtain as follows.

$$(2n+L)(2n+L-1)+2(2n+L)-L(L+1)=0 \quad (21)$$

$$-2\beta(2n+L)+P-2\beta=0 \quad (22)$$

$$-4\alpha(2n+L)+\varepsilon+\beta^2-6\alpha=0 \quad (23)$$

$$4\alpha\beta-Q=0 \quad (24)$$

$$4\alpha^2+S=0 \quad (25)$$

From equation (22)

$$\beta=\frac{P}{4n+2L+2} \quad (26)$$

From equation (25),

$$\alpha=\frac{\sqrt{-S}}{2} \quad (27)$$

The energy equation is obtain using equation (23)

$$\varepsilon=2\alpha(4n+2L+3)-\beta^2 \quad (28)$$

Substituting equations (6), (9),(11),(26) and (27) into equation (28), the energy eigenvalues of HHP is obtain as,

$$E_{nl}=\sqrt{\frac{-\hbar^2 A_2 m_D^3(T)}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2}\right)-\frac{2\mu}{\hbar^2}\left(A_1-A_2+\frac{A_0}{m_D(T)}\right)^2\left(4n+1+\sqrt{(2l+1)^2}\right)^{-2}+\frac{A_0}{2}-A_2 m_D(T) \tag{29}$$

Special cases

Setting $A_0=0$ in equation (29) ,the energy equation for Hellmann potential is obtain:

$$E_{nl}=\sqrt{\frac{-\hbar^2 A_2 m_D^3(T)}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2}\right)-\frac{2\mu}{\hbar^2}\left(A_1-A_2+\frac{A_0}{m_D(T)}\right)^2\left(4n+1+\sqrt{(2l+1)^2}\right)^{-2}-A_2 m_D(T) \tag{30}$$

Setting $A_1=A_2=0$ in equation (29), the energy equation for Hulthén potential is obtain:

$$E_{nl}=-\frac{2\mu}{\hbar^2}\left(+\frac{A_0}{m_D(T)}\right)^2\left(4n+1+\sqrt{(2l+1)^2}\right)^{-2}+\frac{A_0}{2} \tag{31}$$

Setting $A_0=A_2=0$ in equation (29), the obtain energy equation for Coulomb potential is obtain

$$E_{nl}=-\frac{2\mu A_1^2}{\hbar^2}\left(4n+1+\sqrt{(2l+1)^2}\right)^{-2} \tag{32}$$

Setting $A_0=A_1=0$ in equation (29), the energy equation for Yukawa potential is obtain:

$$E_{nl}=\sqrt{\frac{-\hbar^2 A_2 m_D^3(T)}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2}\right)-\frac{2\mu}{\hbar^2}\left(-A_2+\frac{A_0}{m_D(T)}\right)^2\left(4n+1+\sqrt{(2l+1)^2}\right)^{-2}-A_2 m_D(T) \tag{33}$$

We test for the accuracy of the predicted results, using a Chi square function (Ali *et al*., 2020)

$$\chi^2=\frac{1}{k}\sum_{i=1}^{k}\frac{\left(M_i^{\text{Exp.}}-M_i^{Theo.}\right)}{\Delta_i} \tag{34}$$

where k runs over selected samples of heavy mesons, $M_i^{\text{exp.}}$ is the experimental mass of heavy mesons, while M_i^{Th} is the corresponding theoretical prediction. The Δ_i quantity is experimental uncertainty of the masses. Intuitively, Δ_i should be one.

RESULTS AND DISCUSSION

The mass spectra of the heavy mesons such as charmonium and bottomonium that have the quark and antiquark flavor is calculated by applying the following relation (Inyang, *et al*., 2021).

$$M=2m+E_{nl}, \tag{35}$$

where m is heavy quark mass, and E_{nl} is energy eigenvalues.

By substituting equation (29) into equation (35), the mass spectra for HHP is obtain as:

$$M=2m+\sqrt{\frac{-\hbar^2 A_2 m_D^3(T)}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2}\right)-\frac{2\mu}{\hbar^2}\left(A_1-A_2+\frac{A_0}{m_D(T)}\right)^2\left(4n+1+\sqrt{(2l+1)^2}\right)^{-2}+\frac{A_0}{2}-A_2 m_D(T) \tag{36}$$

Table 1: Mass spectra of charmonium in (GeV) (m_c =1.209 GeV, μ = 0.6045 GeV, A_0 = 1.422 GeV, A_1 = 2.949 GeV, A_2 = - 0.009 GeV, $m_D(T)$ = 1.52 GeV, $\hbar$ = 1)

State	Present work	Abu-Shady, 2016	Ciftci,andKisoglu, 2018	Tanabashi*et al.*, 2018
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.525	3.255	3.521	3.525
2P	3.772	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.263	4.269	4.433	4.263
1D	3.770	3.504	3.800	3.770
2D	4.159	-	-	4.159
1F	3.874	-	-	-

Table 2: Mass spectra of bottomonium in (GeV) (m_b =4.823 GeV, μ = 2.4115 GeV, A_0 = - 0.323 GeV, A_1 = 2.110 GeV, A_2 = - 0.031 GeV, $m_D(T)$ = 1.52 GeV, $\hbar$ = 1)

State	Present work	Abu-Shady, 2016	Ciftci, and Kisoglu, 2018	Tanabashi*et al.*, 2018
1S	9.460	9.460	9.462	9.460
2S	10.023	10.023	10.027	10.023
1P	9.898	9.619	9.963	9.899
2P	10.256	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.580	10.567	10.624	10.580
1D	10.164	9.864	10.209	10.164
2D	10.306	-	-	-
1F	10.209	-	-	-

DISCUSSION

The prediction of the mass spectra of heavy mesons such as charmonium and bottomonium for different quantum states using equation (36) are presented in Tables 1 and 2. The free parameters of equation (36) were then obtained by solving two algebraic equations in the case of charmonium and bottomonium, respectively.

For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$, the numerical values of these masses as $m_b = 4.823\,GeV$ and $m_c = 1.209\,GeV$ were adopted (Olive *et al.*, 2014). Then, the corresponding reduced mass are $\mu_b =$ 2.4115 GeV and $\mu_c = 0.6045\,GeV$, respectively. The experimental data were taken from Tanabashi *et al.*, (2018). It is noticed that the prediction of mass spectra of charmonium and bottomonium are in good agreement with experimental data and the work of other researchers, as presented in Tables 1 and 2. In order to test for the accuracy of the predicted results, a Chi square function is used to determine the error between the experimental data and theoretical predicted values. The maximum error in comparison with the experimental data is found to be 0.034 GeV.

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